The Twist Operators on Maniplexes

Ian Douglas,
Isabel Hubard,
Daniel Pellicer

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Maniplex:

You make an n+1 – dimensional one

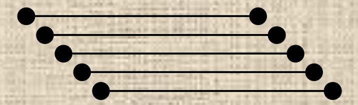
By taking a bunch of n-dimensional ones

And glueing them together along isomorphic n-1-dimensional bits.

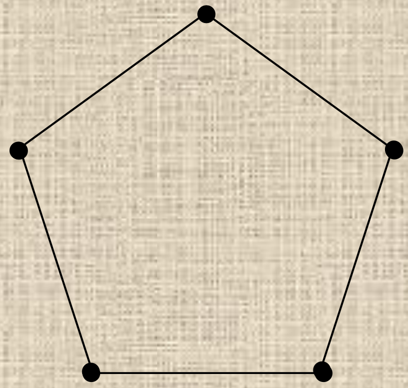
One-dimensional ones are segments.

Maniplex:

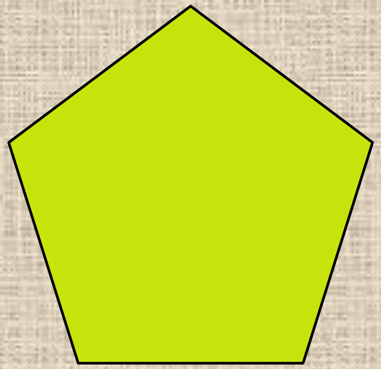
Start with a bunch of segments:



Glue these 1’s together by their vertices (0’s) to make a 2-maniplex:

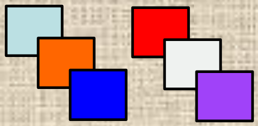


A polygon:

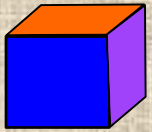


Maniplex:

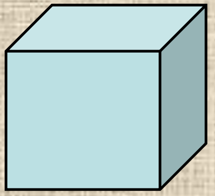
Start with a bunch of squares:



Glue these 2’s together by their edges(1’s) to make a 3-maniplex:



A cube:



Maniplex: M = (Ω, R), where

Ω is a set of things called ‘flags’

$R = [r_0, r_1, r_2, \ldots, r_n]$

Each r_i is an involution on Ω.

Each r_i commutes with all others except perhaps r_{i+1} and r_{i-1}.

The group C generated by R is transitive on Ω.

Faces and Facets

Let $C_i = \langle R \backslash r_i \rangle$.

Orbits under C_i are i-faces.

0-faces are ‘vertices’.

1-faces are ‘edges’

n-faces are ‘facets’

facets of facets are ‘subfacets’

$(R \backslash \{r_n, r_{n-1}\})$

Colors and orientability

Pick a flag and color it red.

Recursively colour neighbors of reds white and neighbours of whites red.

R = red flags, W = white flags.

On of two things will happen:

1: $R \cap W =$ Æ M is *orientable*

2: All flags will be both red and white. M is *non-orientable.*

Symmetry

A *symmetry* of M = (Ω, R) is a permutation σ of Ω which respects R:

For f in Ω and r_i in R, $(r_i f)\sigma = r_i (f\sigma)$

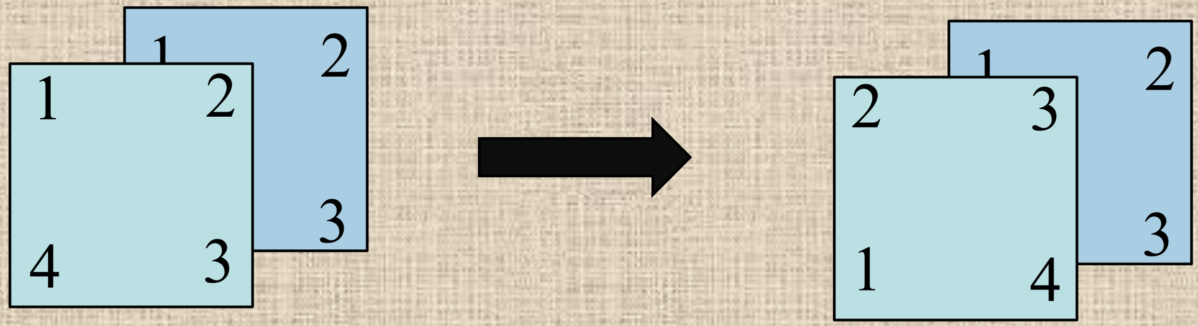
Group of symmetries is Aut(M)

M is *rotary* if Aut(M) is transitive on R
M is *reflexible* if Aut(M) is transitive on Ω.

So a non-orientable rotary maniplex must be reflexible.

Rotary but not reflexible is *chiral.*

The Twist



This amounts to replacing $r_3 f$ by:

$r_0 r_1 r_3 f$ if f is Red

$r_1 r_0 r_3 f$ if f is White

The new one is T(M).

The Twist

$T_1(M) = T(M)$

$T_2(M) = T(T_1(M))$

If M is reflexible, $T_j(M)$ is the mirror image of $T_{-j}(M)$.

If M is reflexible, $T_j(M)$ is usually chiral.

The Twist

For instance, consider the 4-cube C, a reflexible maniplex of tpe {4, 3, 3}

T(C) is a chiral maniplex of type {4, 3, 8}

Almost all chiral 4-maniplexes are formed as $T_j(M)$ for some j and some reflexible M.

The General Twist

For higher dimensions, we modify the construction only slightly

If $M = (\Omega, [r_0, r_1, r_2, \ldots, r_n])$ is an orientable maniplex,
let w be a suitable element of
$\langle r_0, r_1, r_2, \ldots, r_{n-2} \rangle$,

and define s_n by

$s_n f = w r_n f$ if f is Red
$s_n f = w^{-1} r_n f$ if f is White

The General Twist

It is easy to check that s_n is an involution, regardless of w.

For $T_w(M) = (\Omega, [r_0, r_1, r_2, \ldots, r_{n-1}, s_n])$ to be a maniplex,

we need all $(wr_i)^2$ for $0 \leq i \leq n-2$ to be trivial

When is $T_w(M)$ reflexible?

Suppose that M is reflexible.

Choose an arbitrary root flag I and label each flag f of M with the unique element of C that sends I to f.

Then C acts on the right as symmetries of M:

The function which sends g to gr_i is a symmetry of M.

When is $T_w(M)$ reflexible?

Suppose that $T_w(M)$ is also reflexible.

Let α_i be the symmetry of $T_w(M)$ which sends the flag I to its i-neighbor.

Consider a flag of the form:

$$g = h_{k+1}(r_n h_k)\,(r_n h_{k-1})\,(r_n h_{k-2})\;.\;.\;.\;(r_n h_0)$$

Define P(g) to be

$$g = h_{k+1}(w^{\pm 2} r_n h_k)\,(w^{\pm 2} r_n h_{k-1})\,(w^{\pm 2} r_n h_{k-2})\;.\;.\;.\;(w^{\pm 2} r_n h_0),$$

Where the signs depend on the parity of the part of the product that follow. .

When is $T_w(M)$ reflexible?

Then the flag $g\alpha_i = P(g)r_i$ for $i = 0, 1, 2, . . . , n-1$, and $g\alpha_n = P(g)wr_n$.

These α_i are defined only if P(g) is well-defined.

This happens only if for each defining word *d* in the presentation of the group, P(*d*) is also equal to the identity.