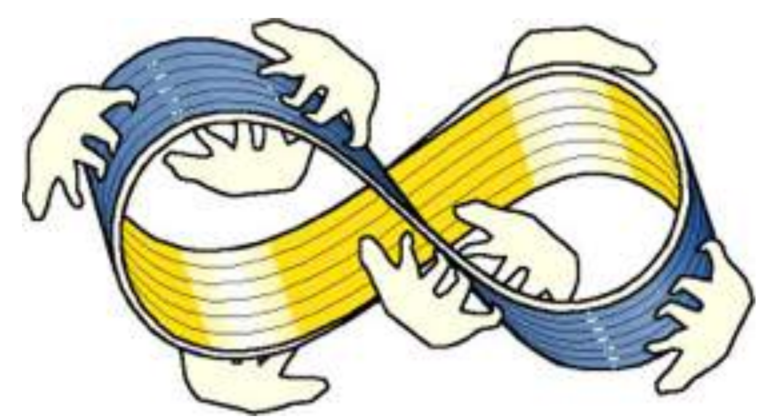
Polytopes with Preassigned Automorphism Groups

Egon Schulte and Gordon Williams

UNIVERSITY OF ALASKA FAIRBANKS



Or:
How I Finally Got My
Schulte Number
Down to 1

A Brief Outline

- On collaborating with Egon
- Our problem
- Our solution for abstract polytopes
- Our method for constructing convex realizations

Our First Collaboration

ON THE IMBEDDABILITY OF A CLASS OF GENERALIZED POLYHEDRA

Degree 1

Schulte ——— Williams

Our First Collaboration

~~*ON THE IMBEDDABILITY OF A CLASS OF GENERALIZED POLYHEDRA*~~

Schulte —— Degree 1 —— Williams

Our First Collaboration

~~*ON THE IMBEDDABILITY OF A CLASS OF GENERALIZED POLYHEDRA*~~

Degree ∞

Schulte

Williams

Subsequent Attempts

Schulte

Williams

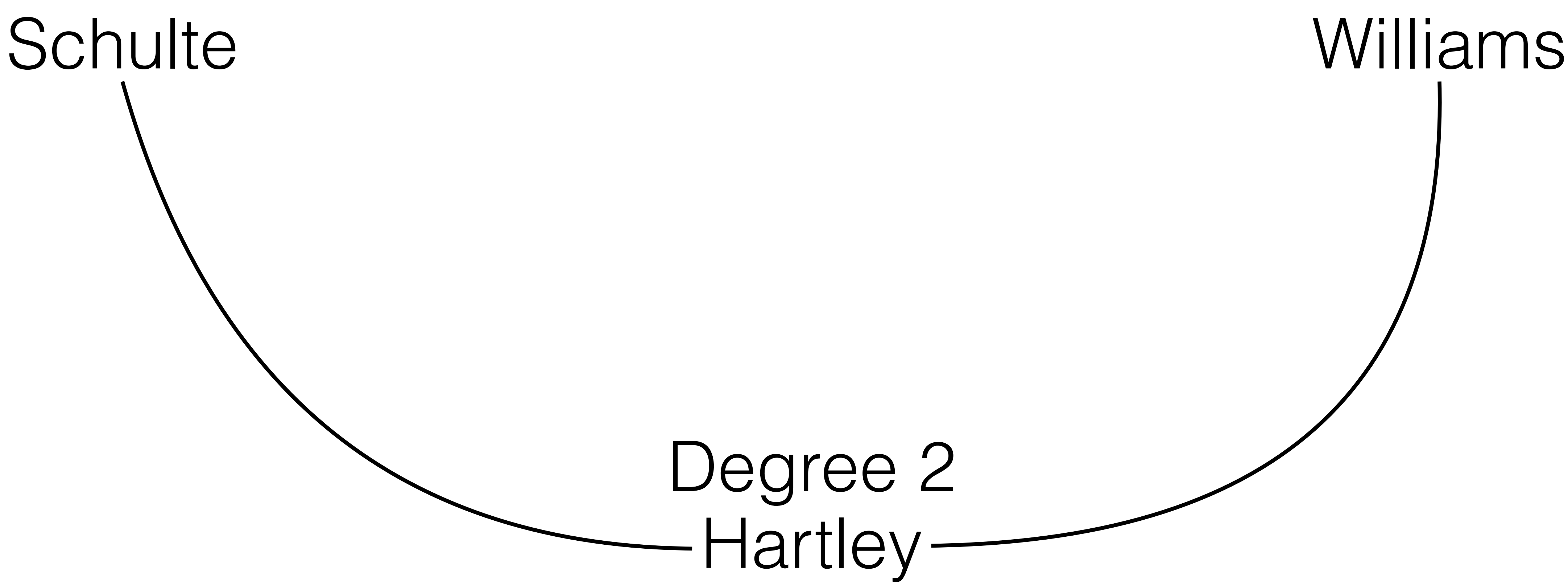
Subsequent Attempts

Schulte Williams

Degree 4

Danzer — Klee — Mihalisin

Subsequent Attempts



Subsequent Attempts

Schulte Williams

Degree 1!

Kaleidoscope

Our Problem

Which finite groups arise as the automorphism groups of abstract polytopes?

Our Answer

Our Answer

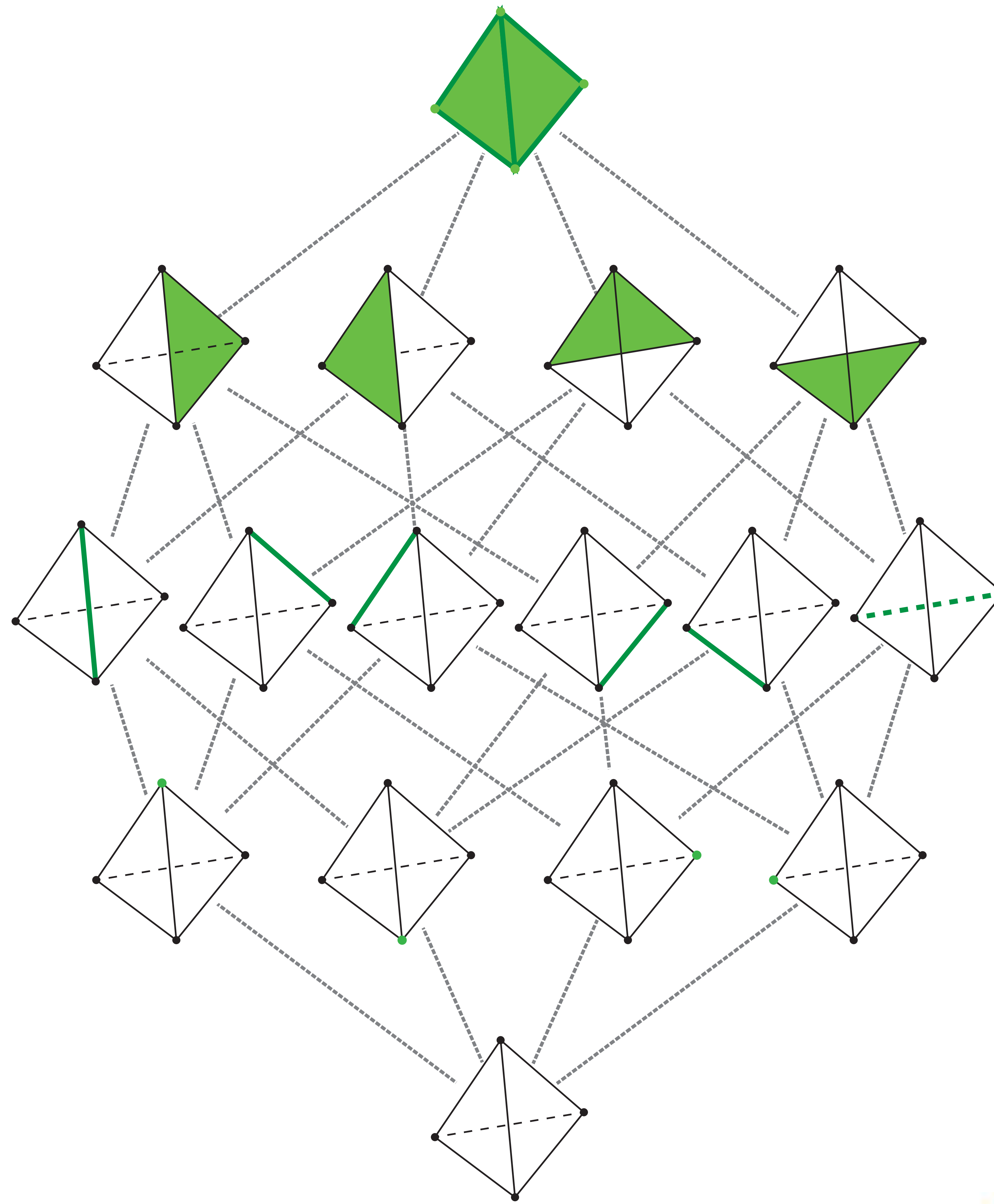
All of them!

Theorem: Every finite group may be realized as the automorphism group of a finite spherical abstract polytope. Moreover, each such polytope admits a realization as the boundary complex of a convex polytope.

Posets as Polytopes

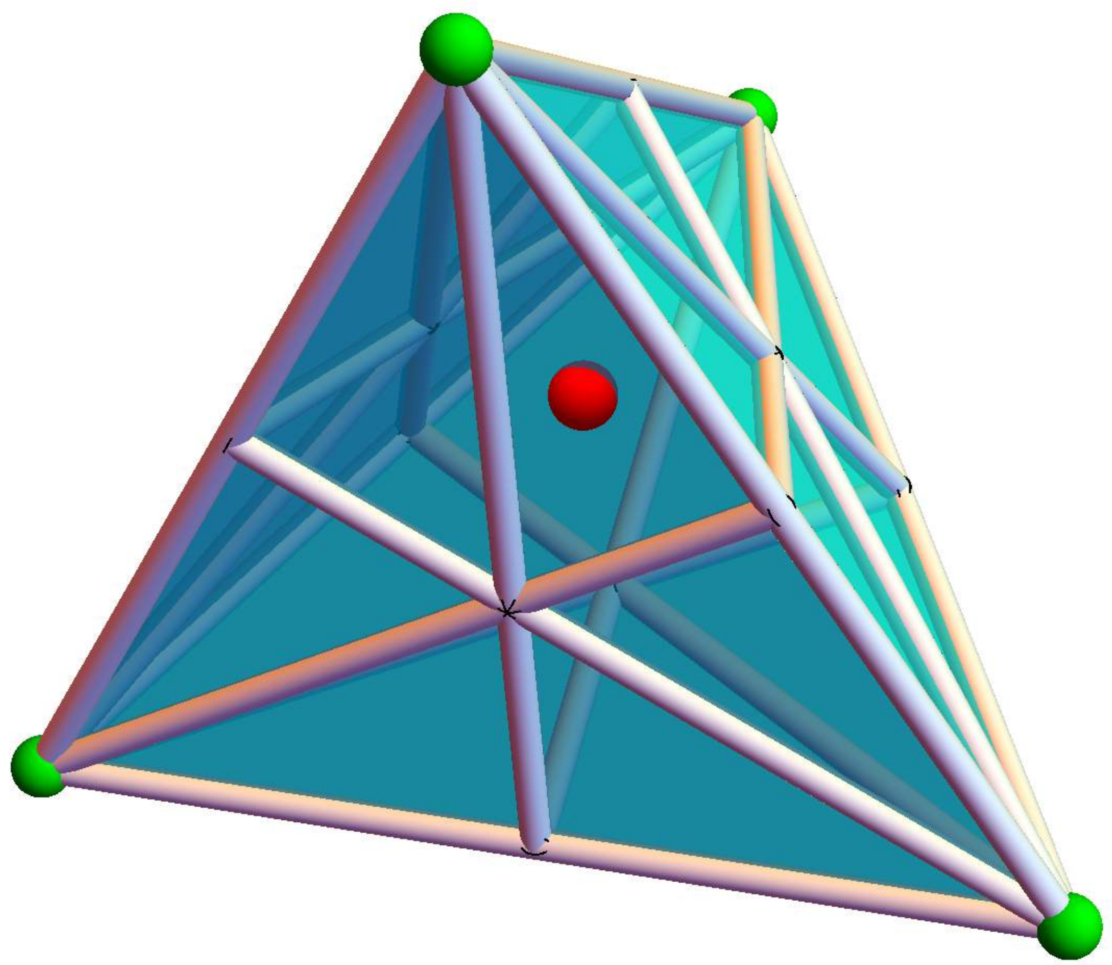
Known as ***abstract polytopes***:

- Least and greatest face
- Maximal chains
- Diamond condition
- Strongly flag connected



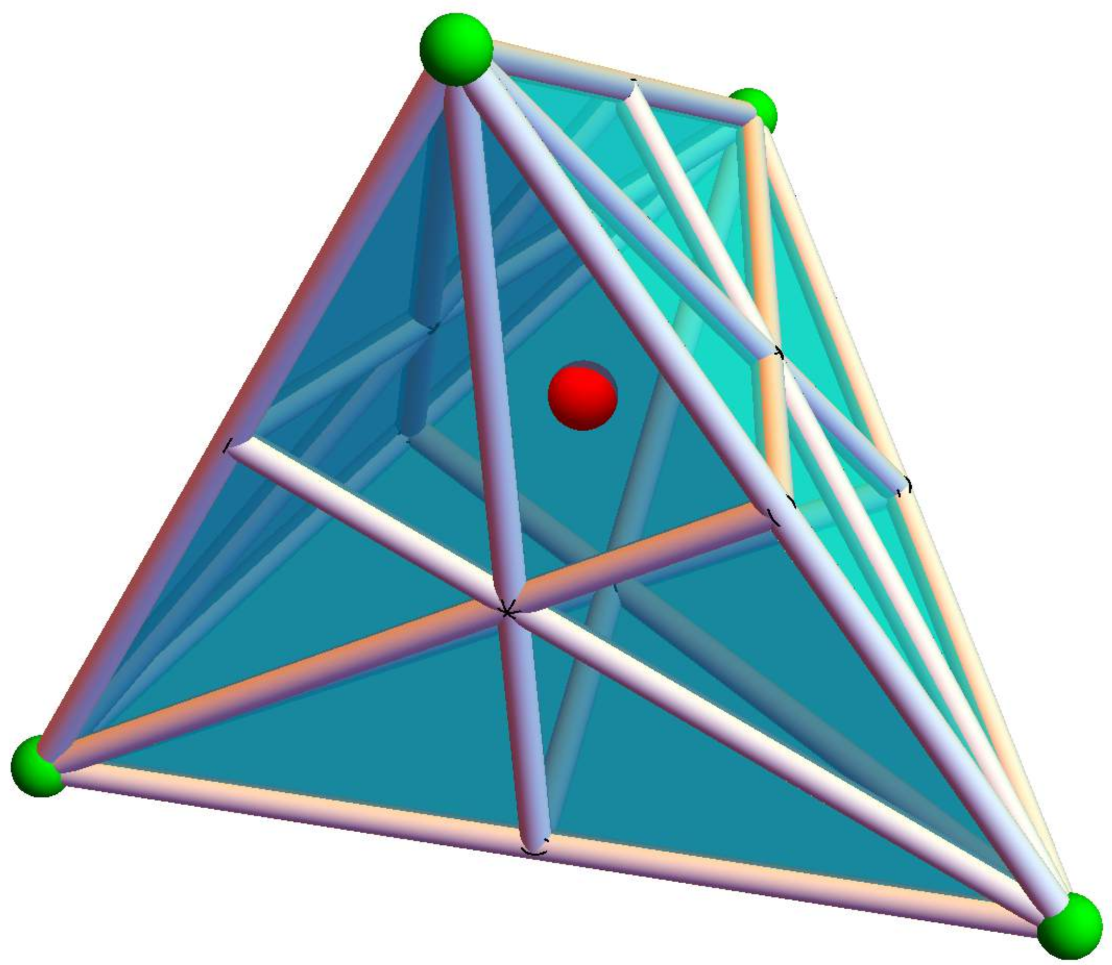
The Big Idea

- Complicate the surface of a suitable convex polytope:



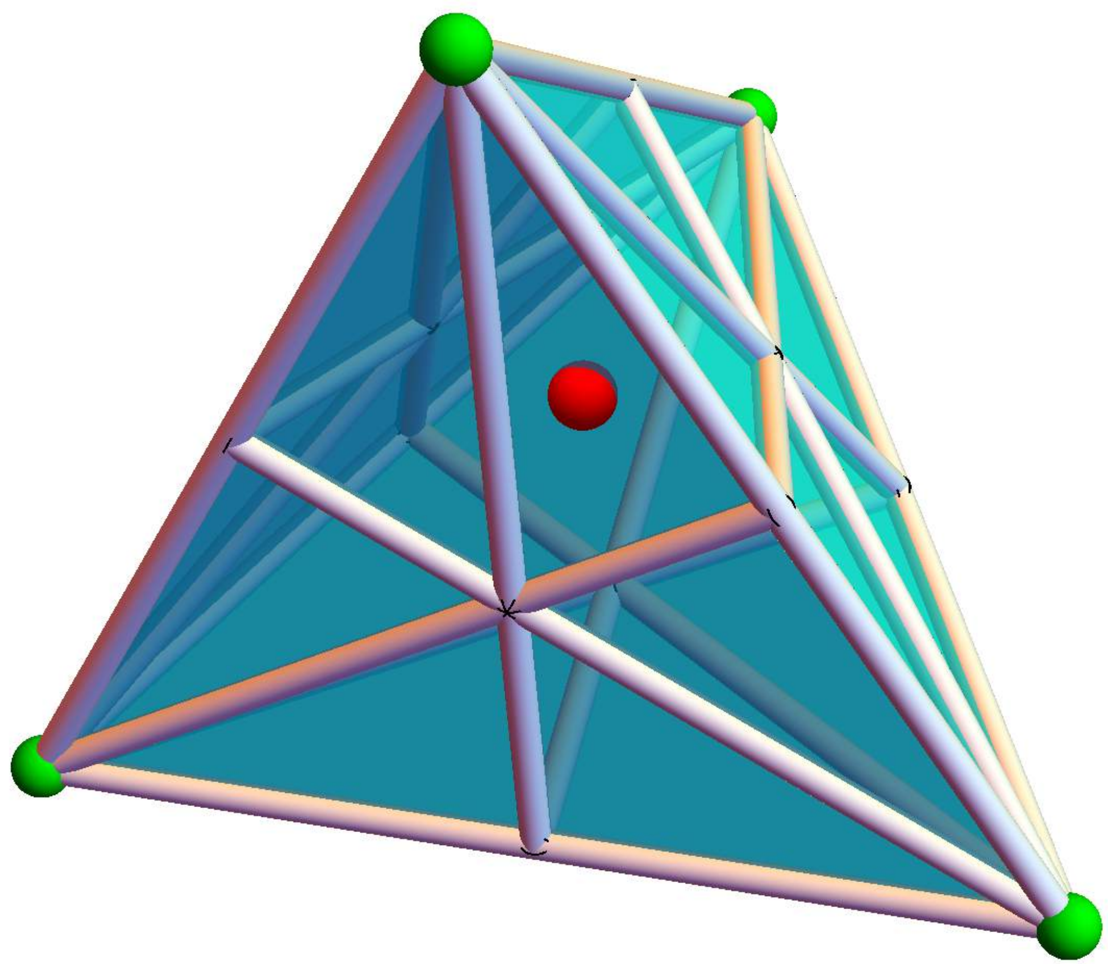
The Big Idea

- Complicate the surface of a suitable convex polytope:
- Treat the desired group G as a subgroup of S_n



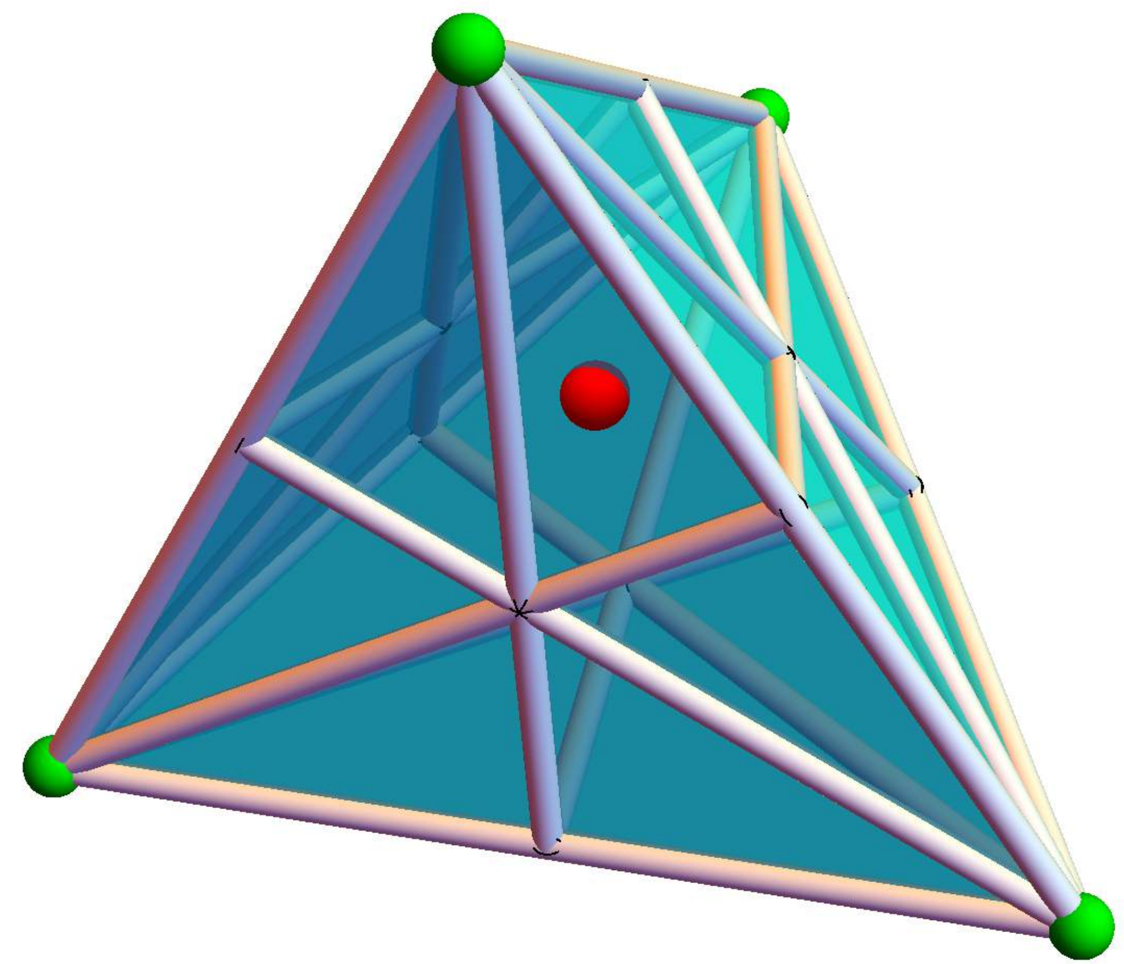
The Big Idea

- Complicate the surface of a suitable convex polytope:
- Treat the desired group G as a subgroup of S_n
    - Consider the orbit under G of a point on the simplex



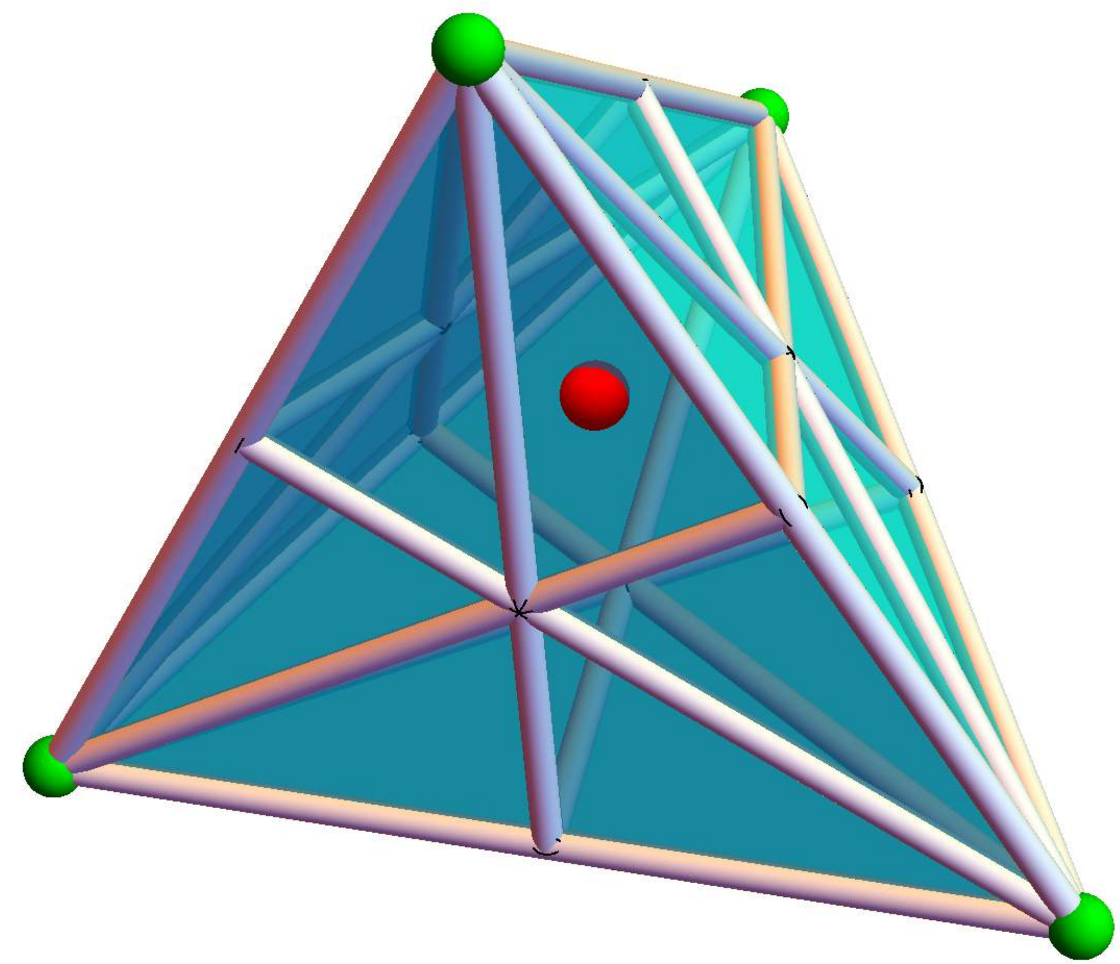
The Big Idea

- Complicate the surface of a suitable convex polytope:
- Treat the desired group G as a subgroup of S_n
  - Consider the orbit under G of a point on the simplex
  - Take the convex hull Q

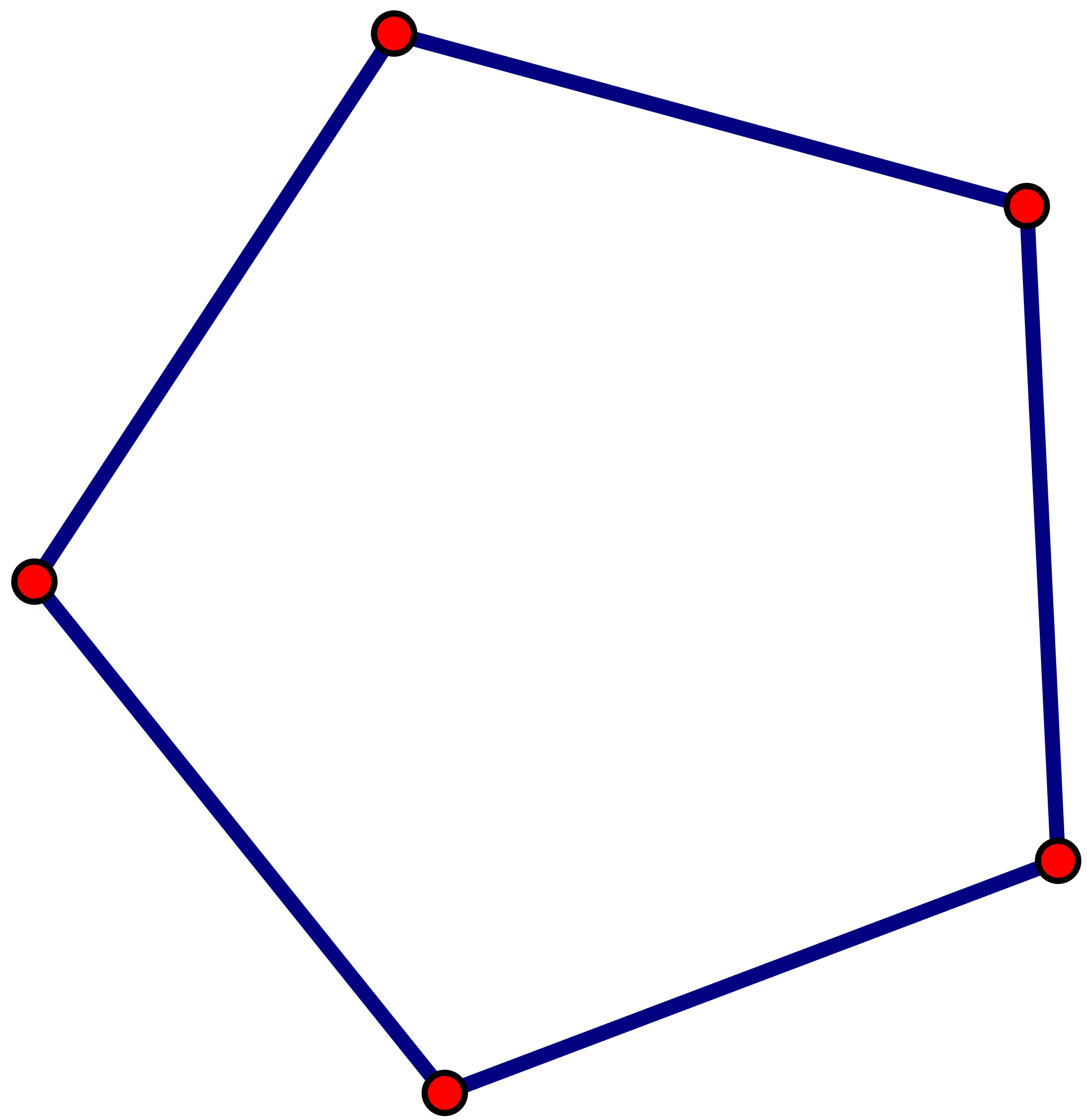


The Big Idea

- Complicate the surface of a suitable convex polytope:
- Treat the desired group G as a subgroup of S_n
  - Consider the orbit under G of a point on the simplex
  - Take the convex hull Q
  - Glue stuff onto Q, breaking symmetry



Exceptions



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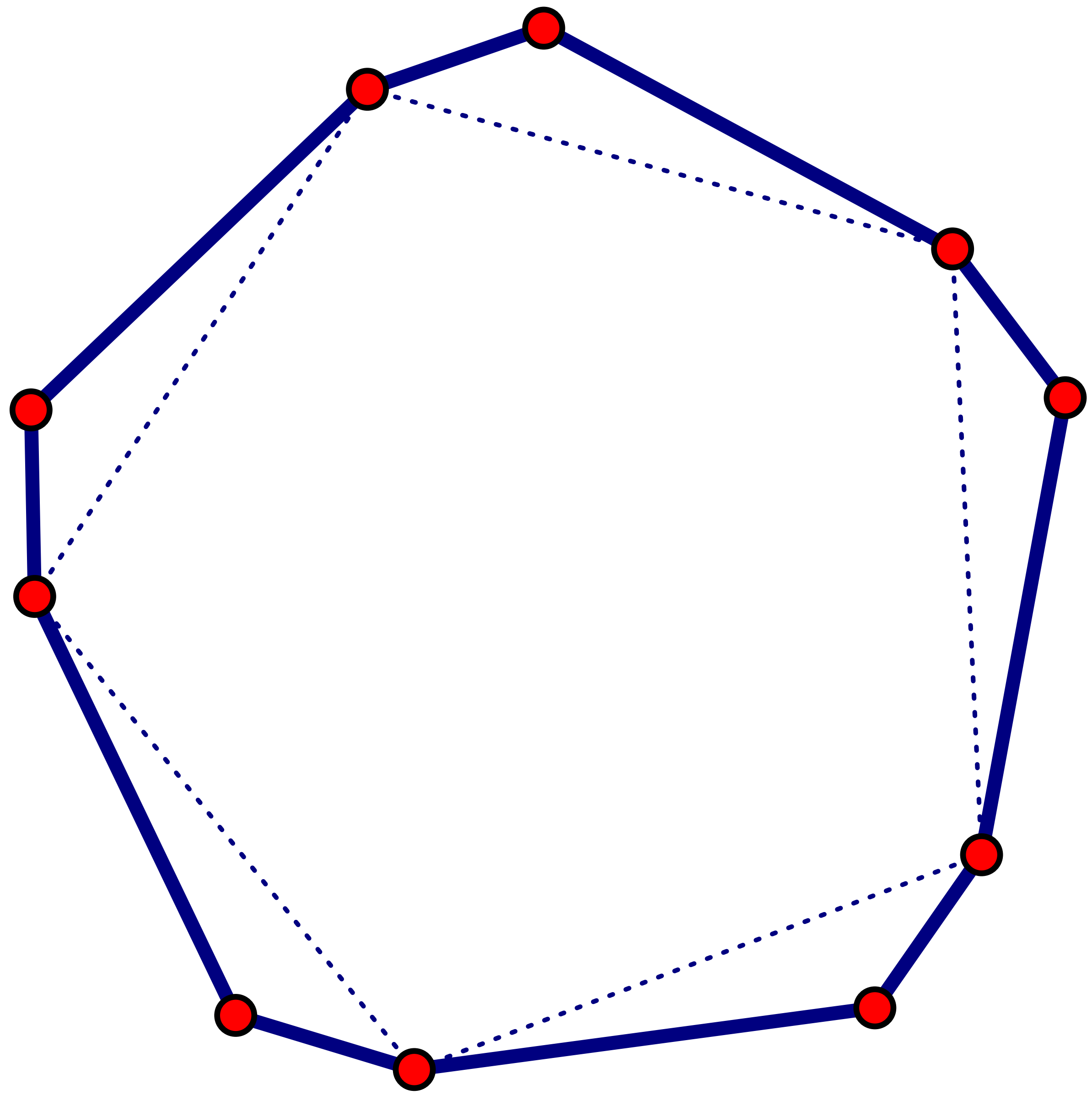
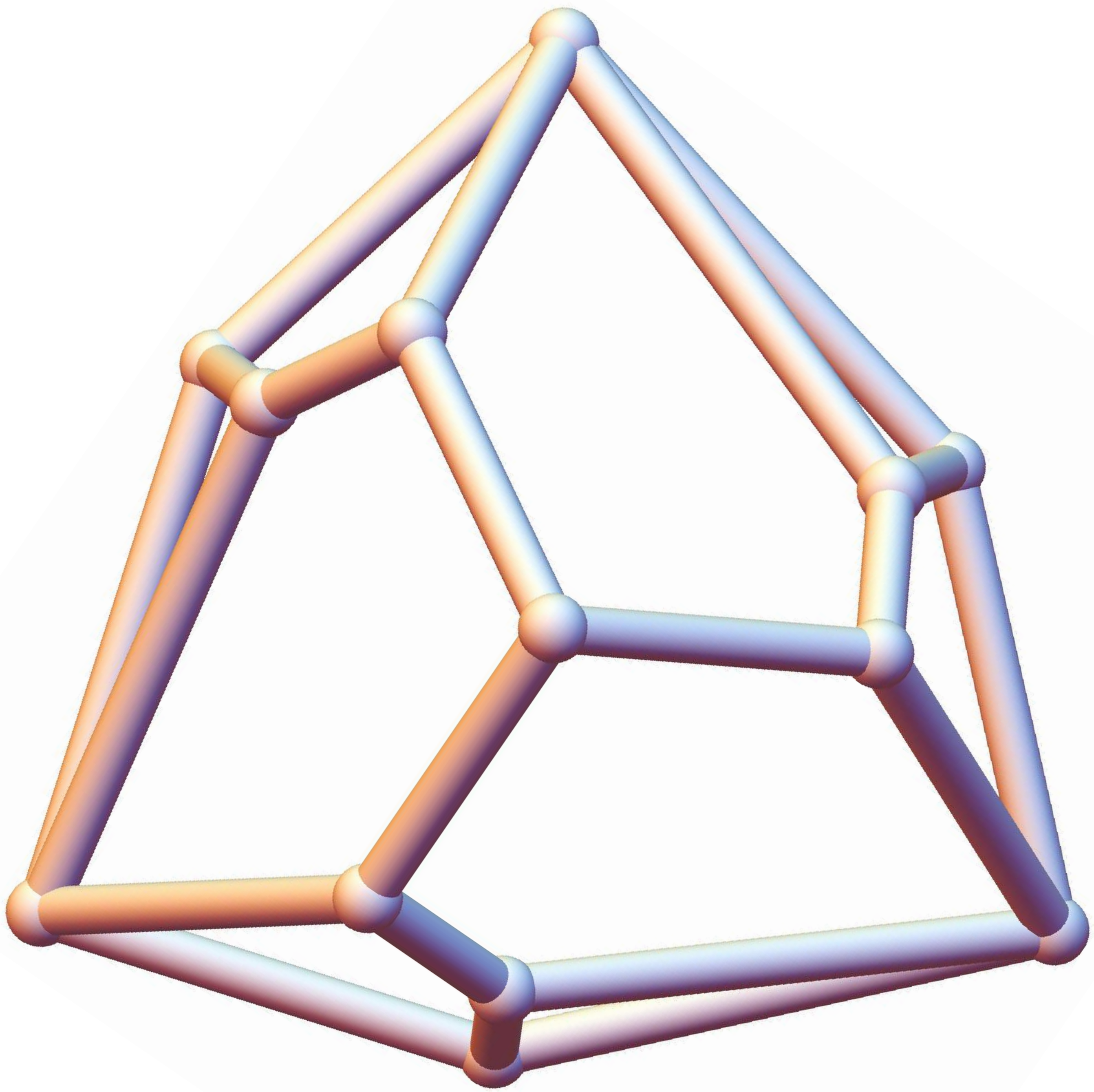
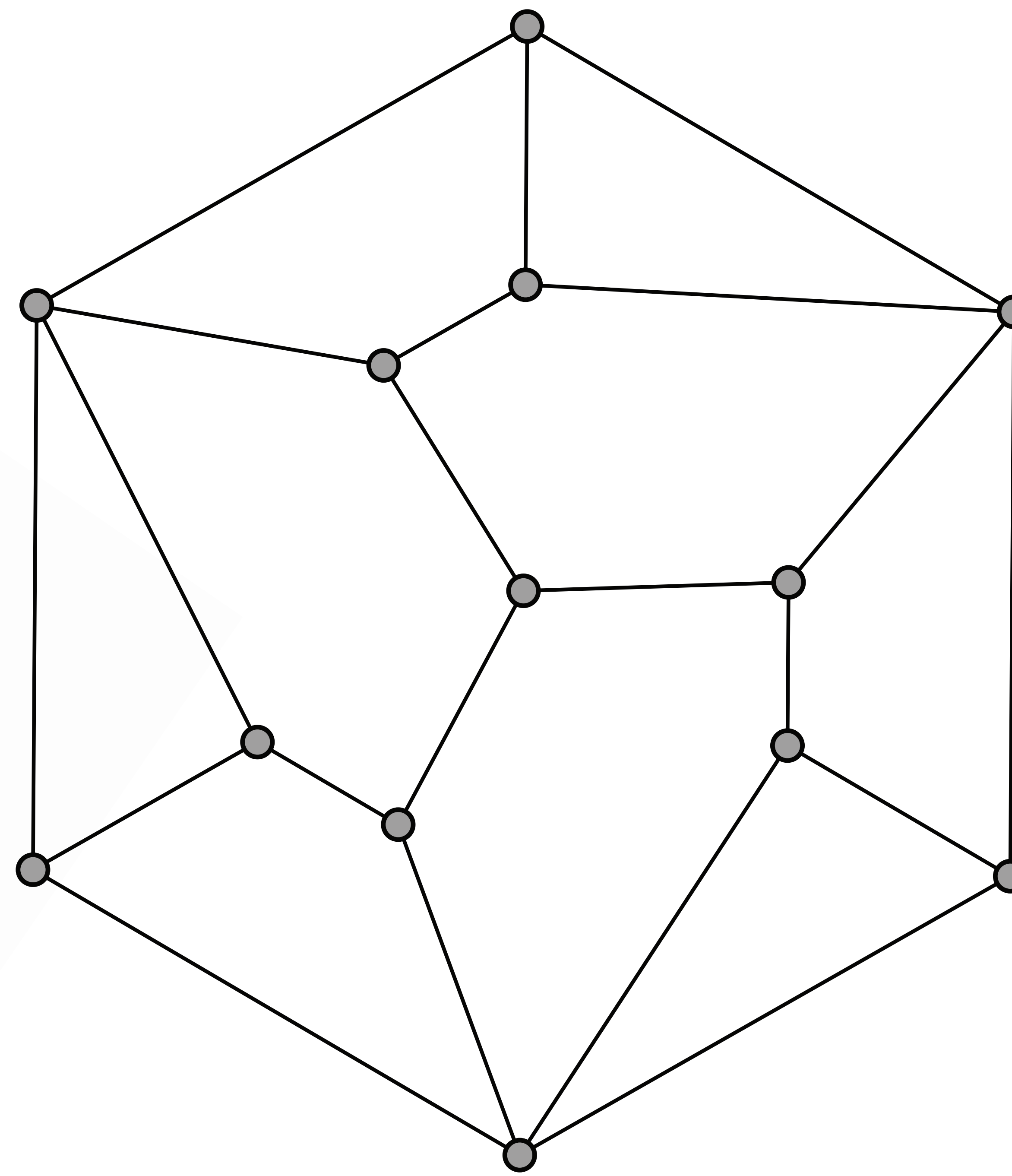


Solution for C_2

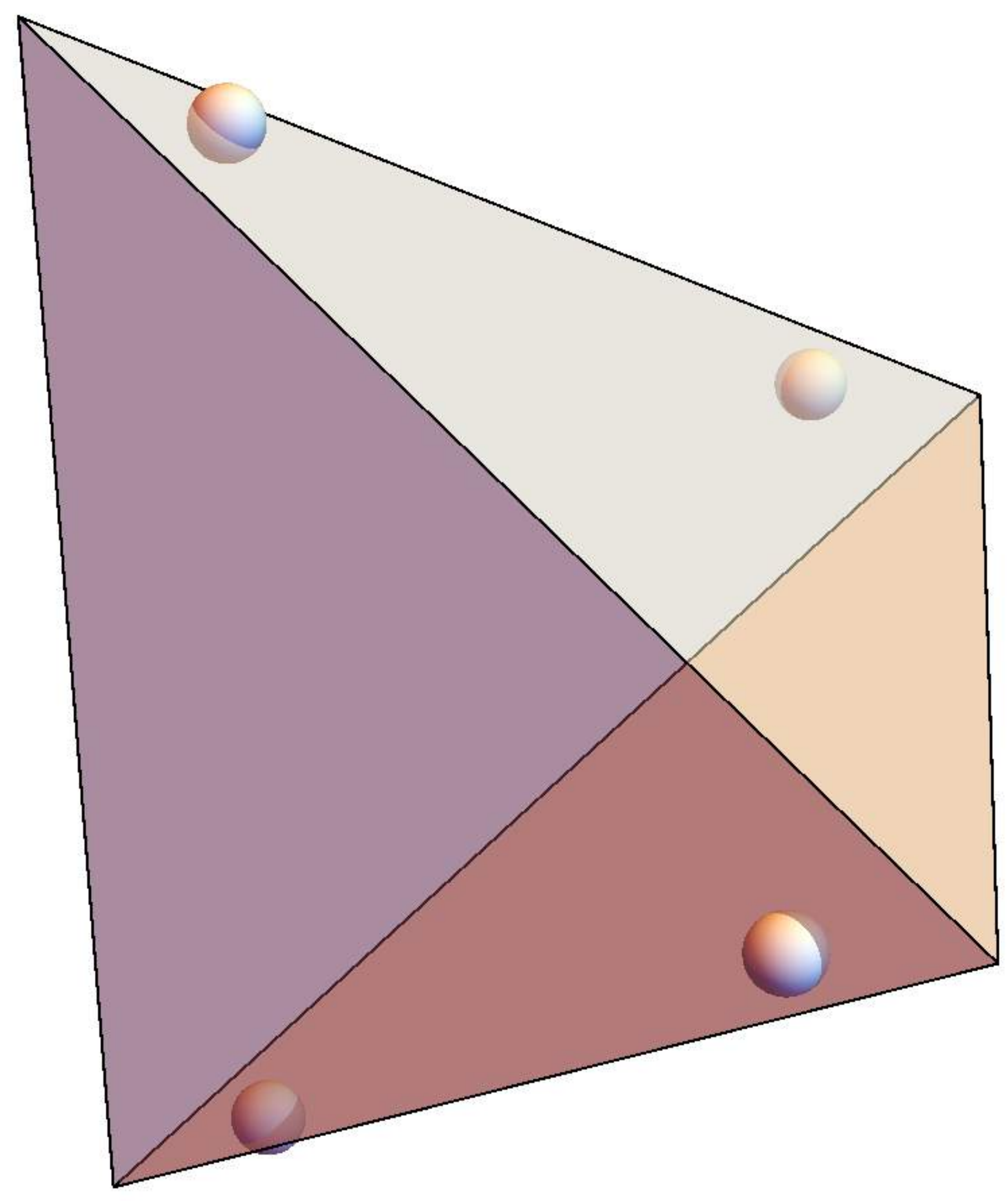
Solution for C_3





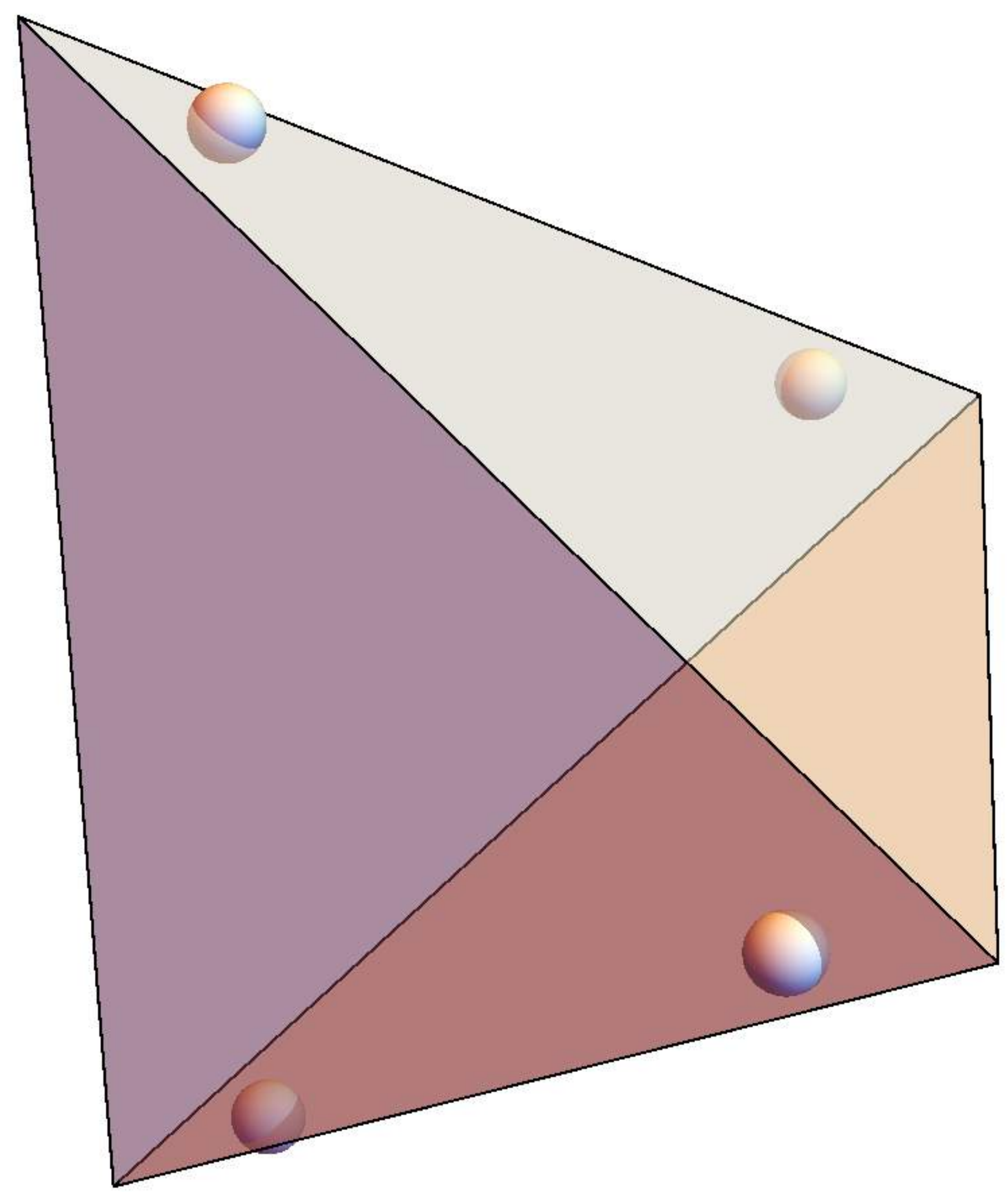
Step 1

- Let S_n be the smallest symmetric group containing G



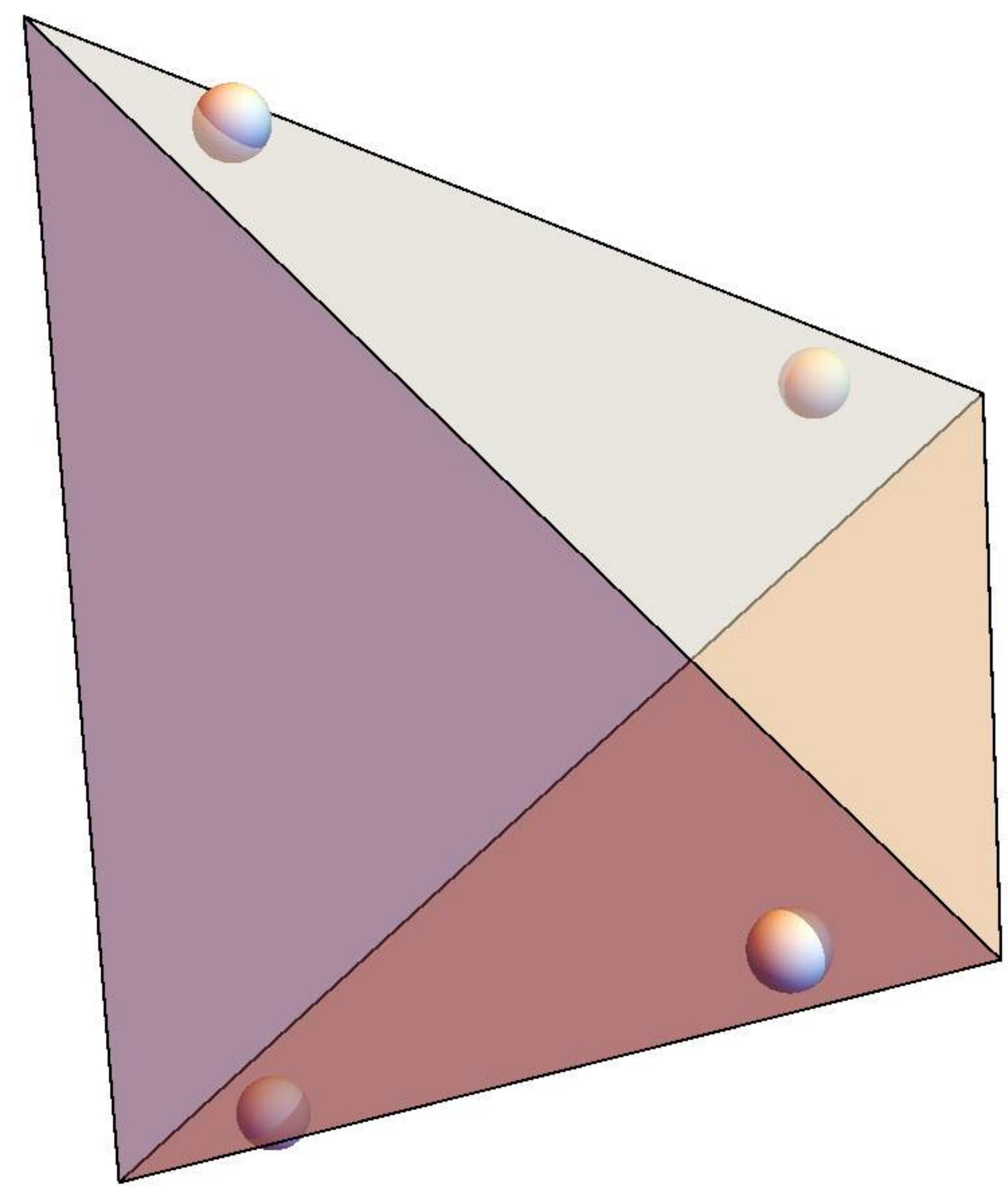
Step 1

- Let S_n be the smallest symmetric group containing G
- Pick a point p inside a simplex of the barycentric subdivision of the boundary complex of $\Delta_{n\text{-}1}$



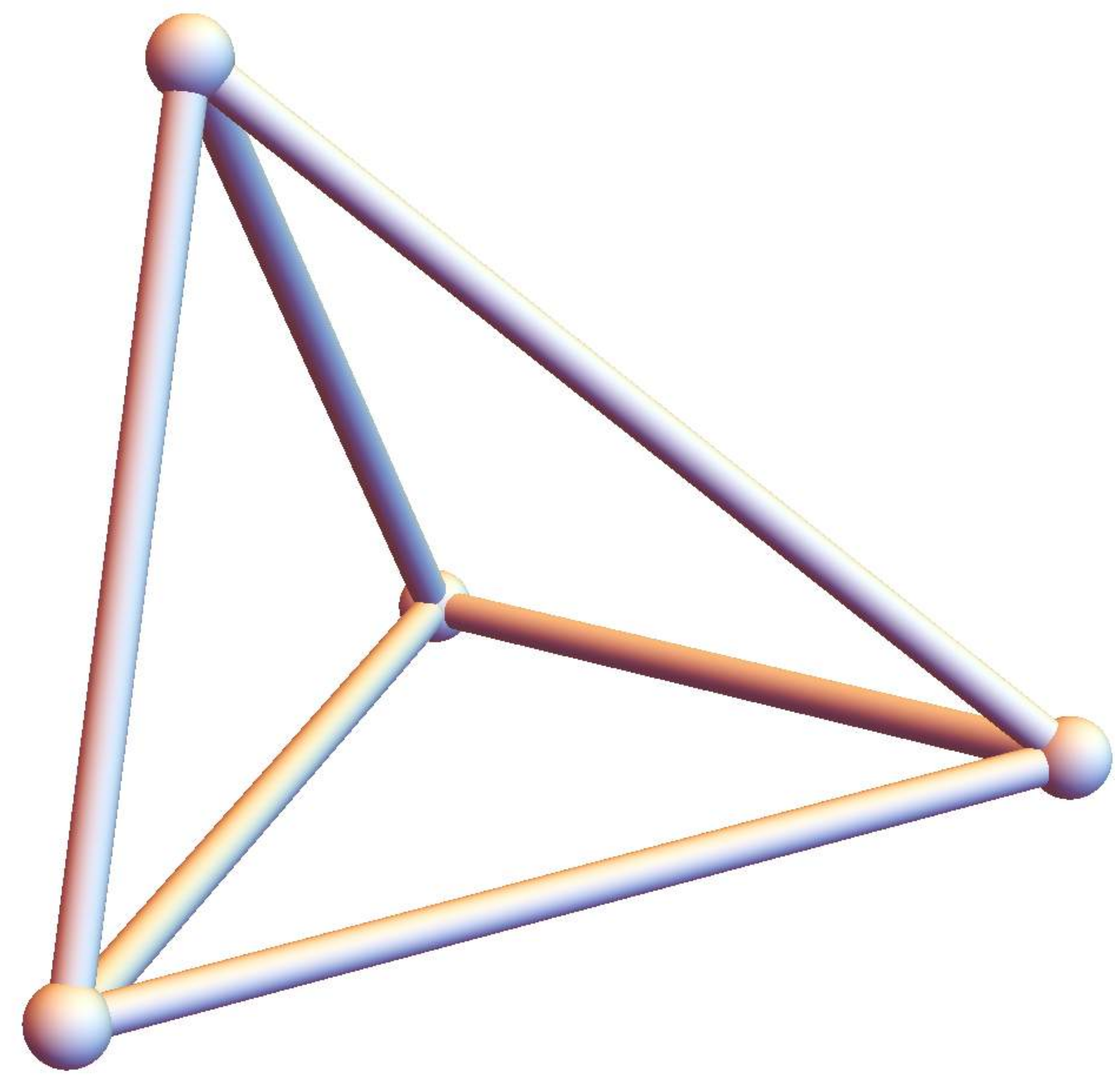
Step 1

- Let S_n be the smallest symmetric group containing G
- Pick a point p inside a simplex of the barycentric subdivision of the boundary complex of Δ_{n-1}
- Construct orbit of p under action of G



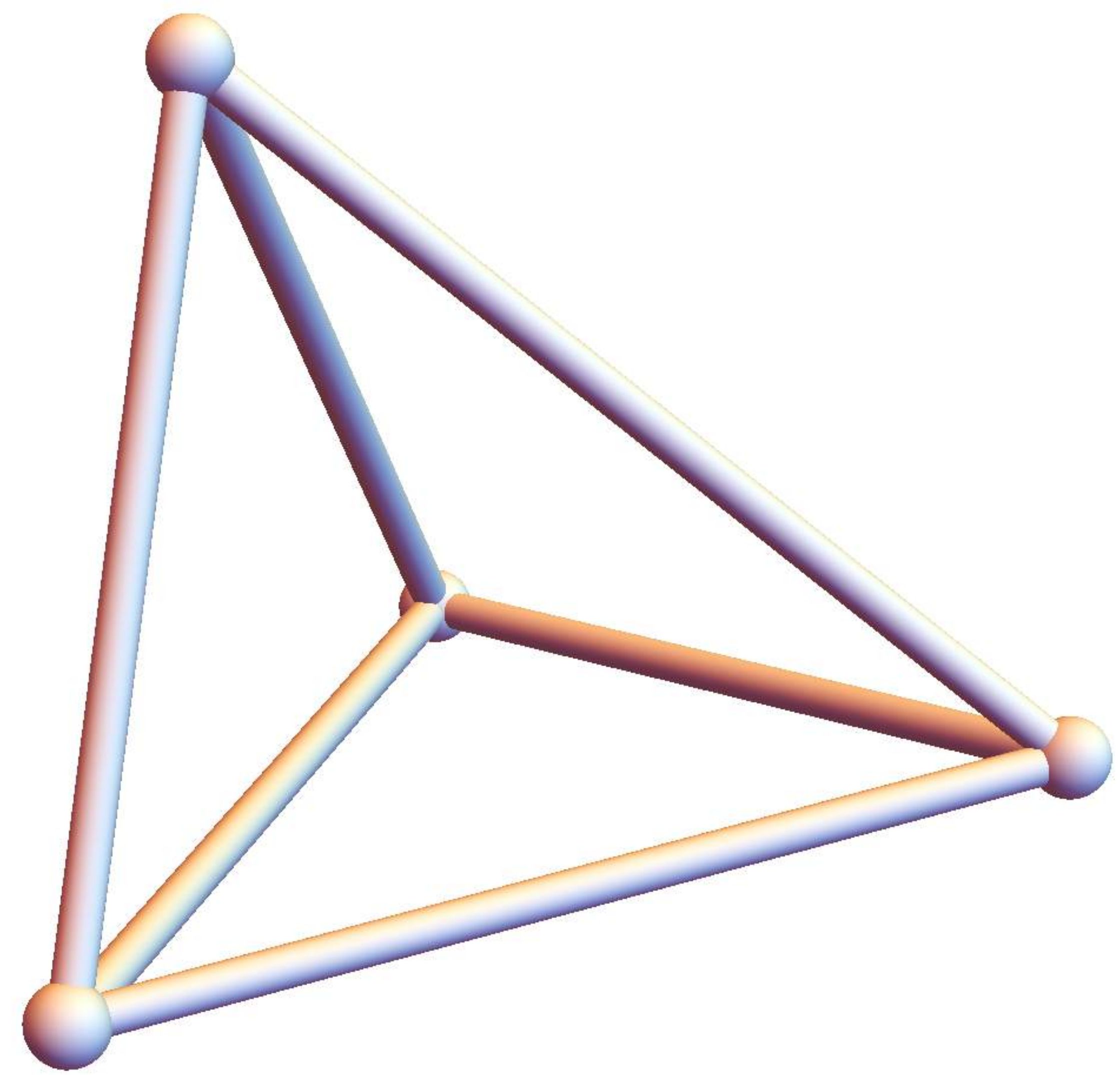
Step 2

- Take the convex hull of this new set of points.



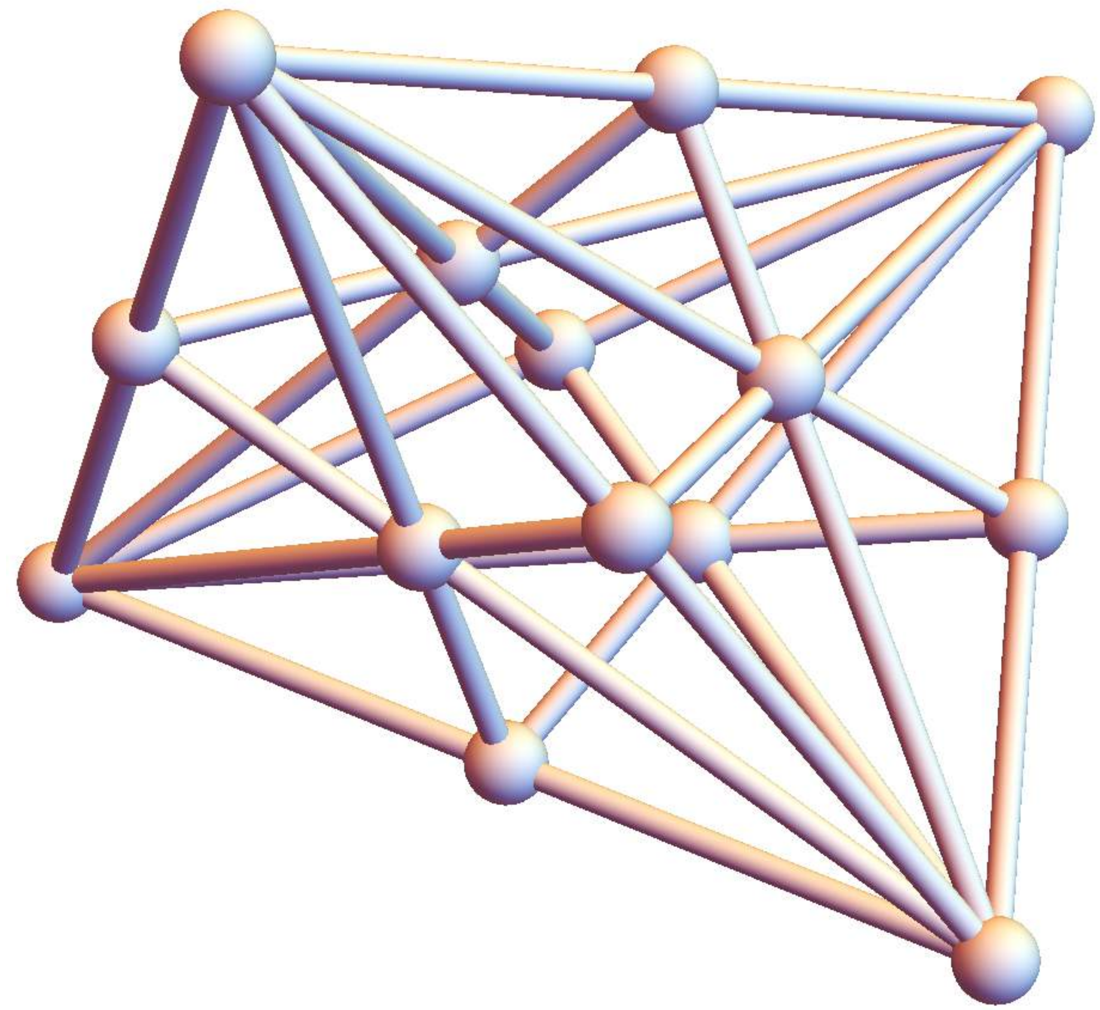
Step 2

- Take the convex hull of this new set of points.
- If lucky, done.



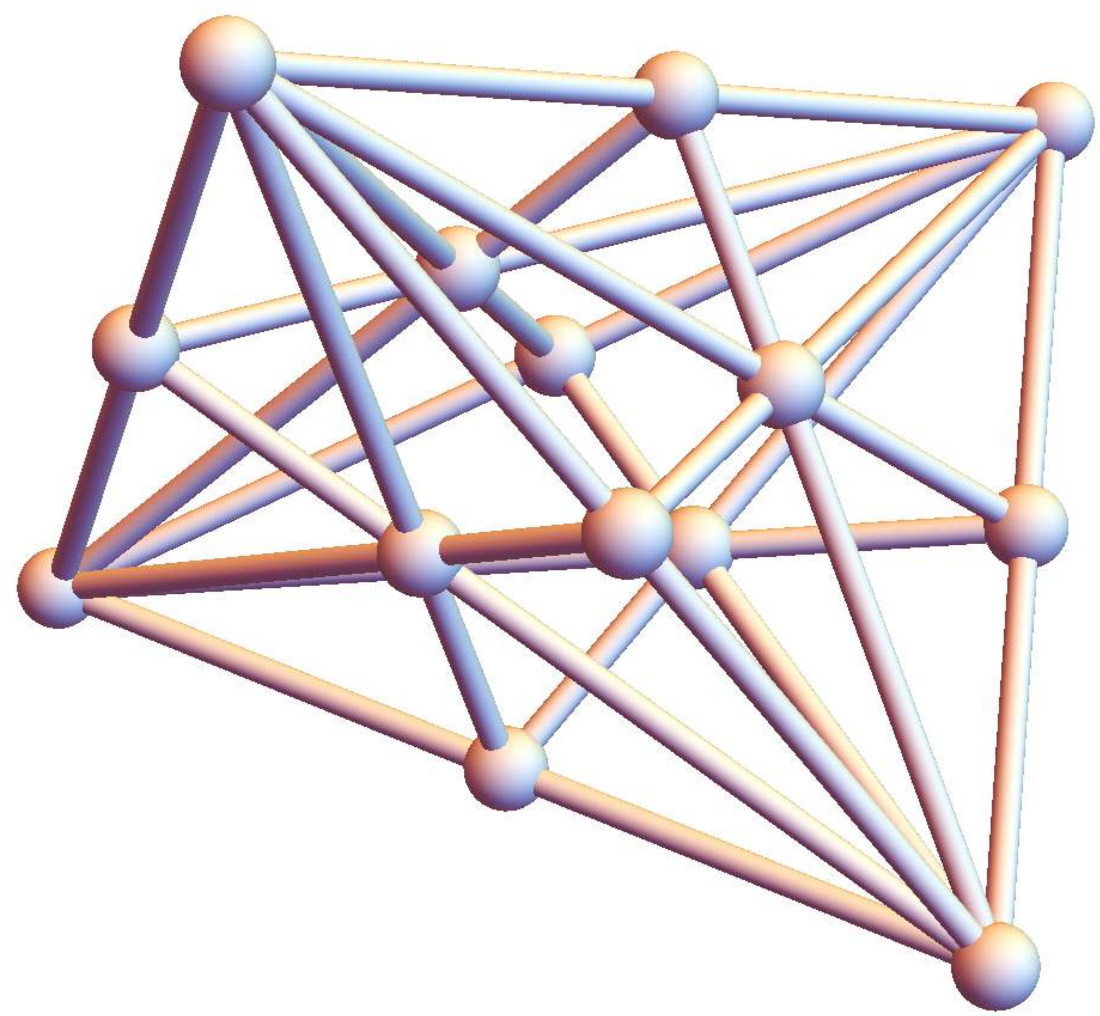
Step 3

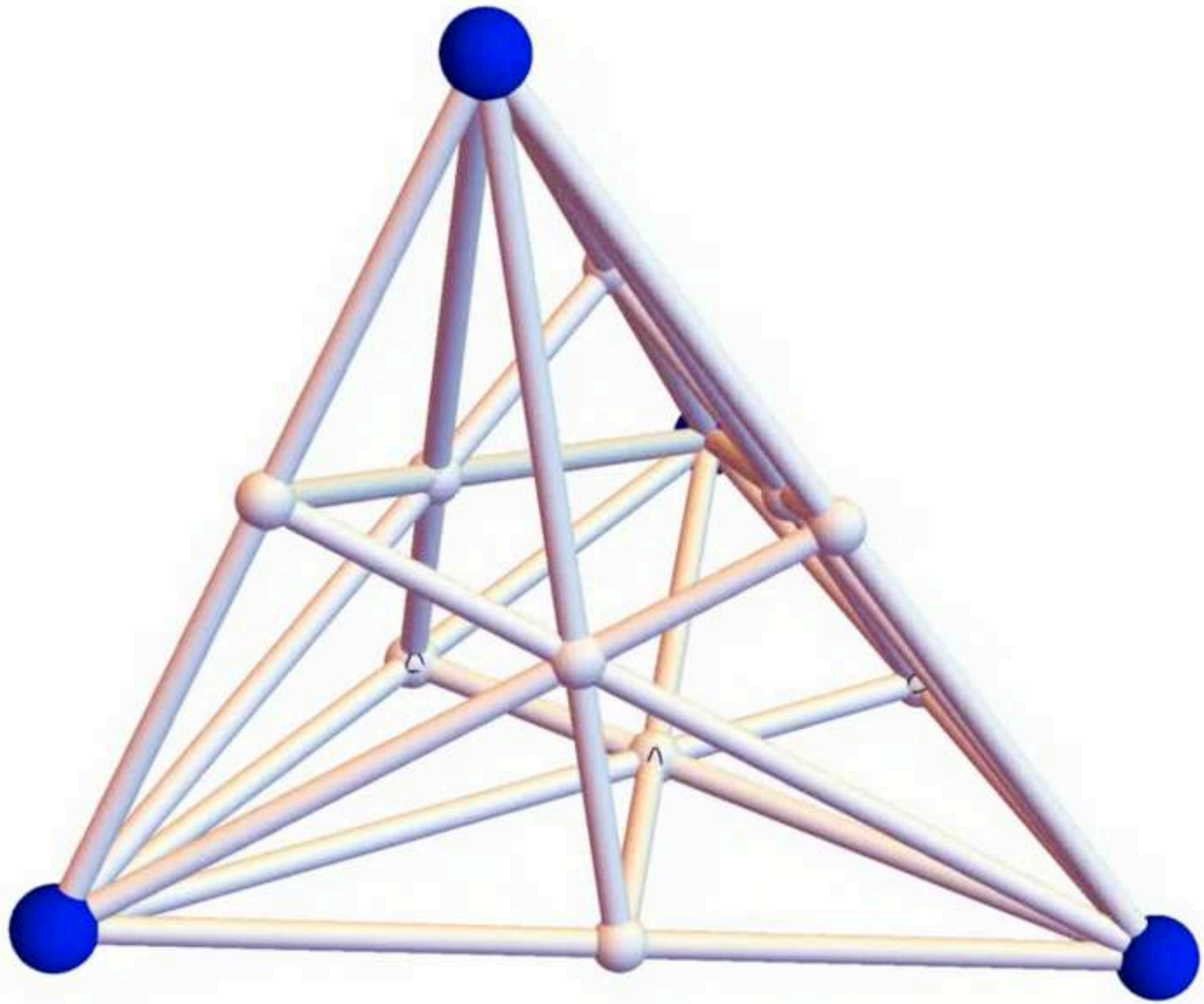
- Take the barycentric subdivision of the boundary complex.

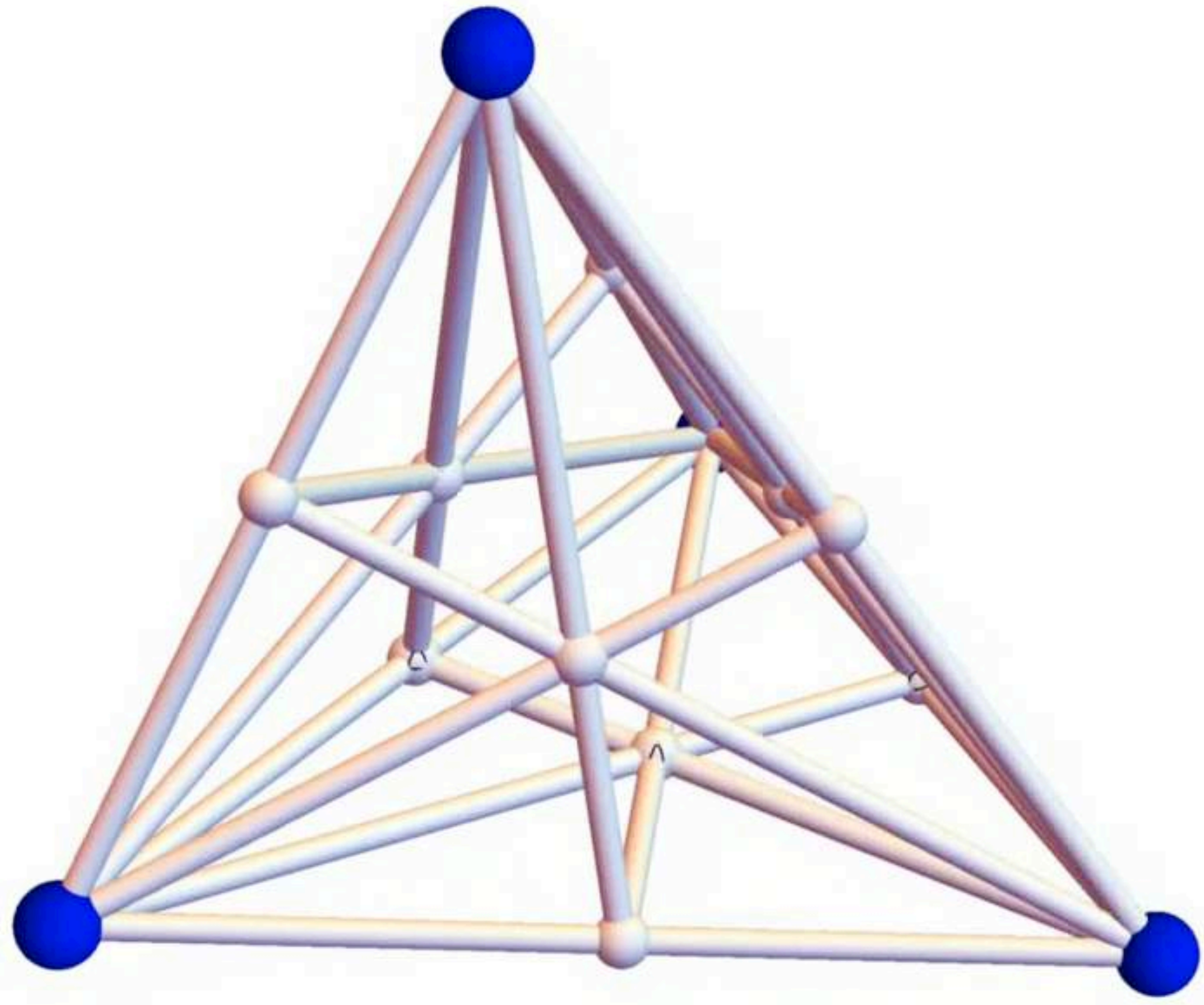


Step 3

- Take the barycentric subdivision of the boundary complex.
- Note that we way construct a convex polytope geometrically of this combinatorial type.

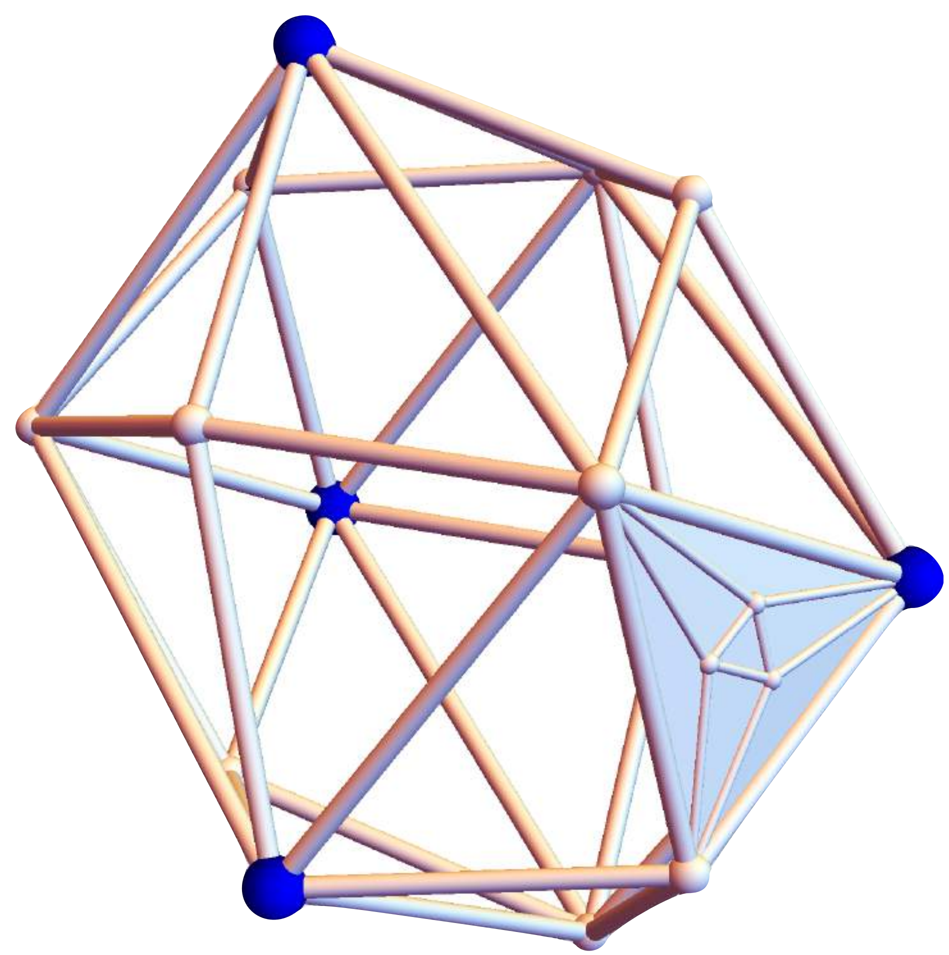






Step 4

- Glue crosspolytopes into the cells of the barycentric subdivision



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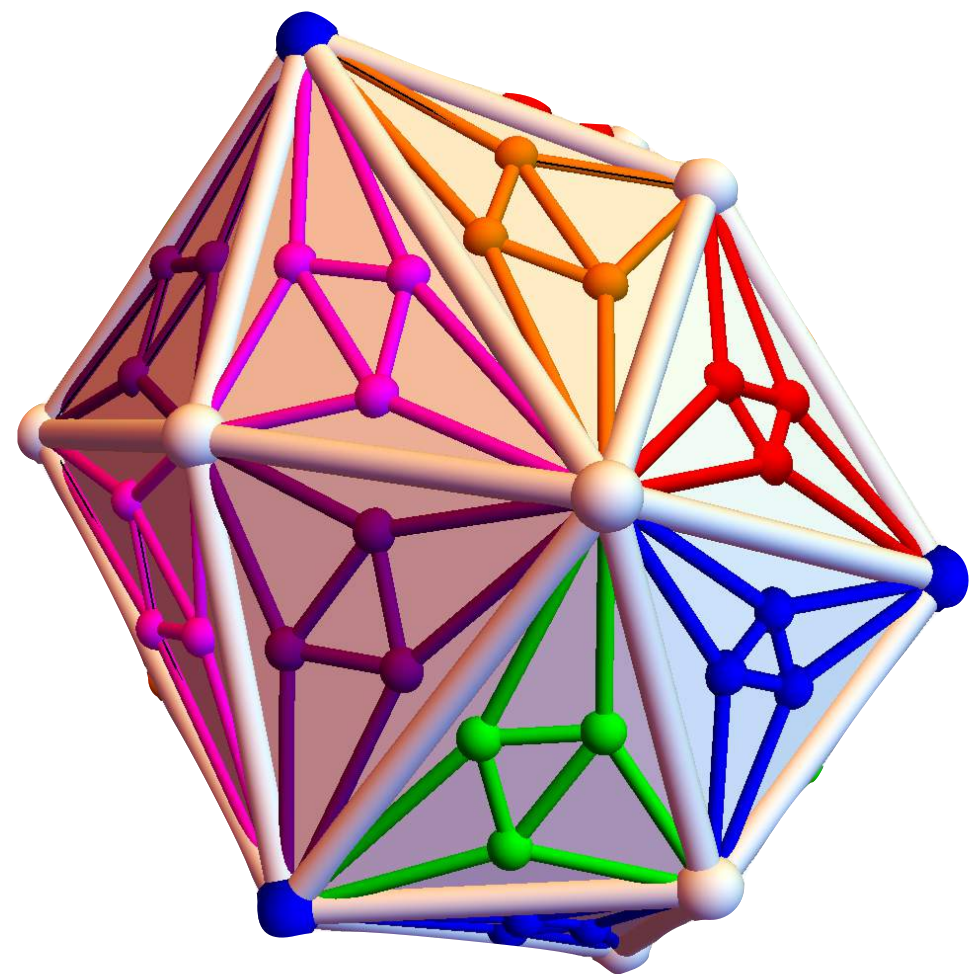
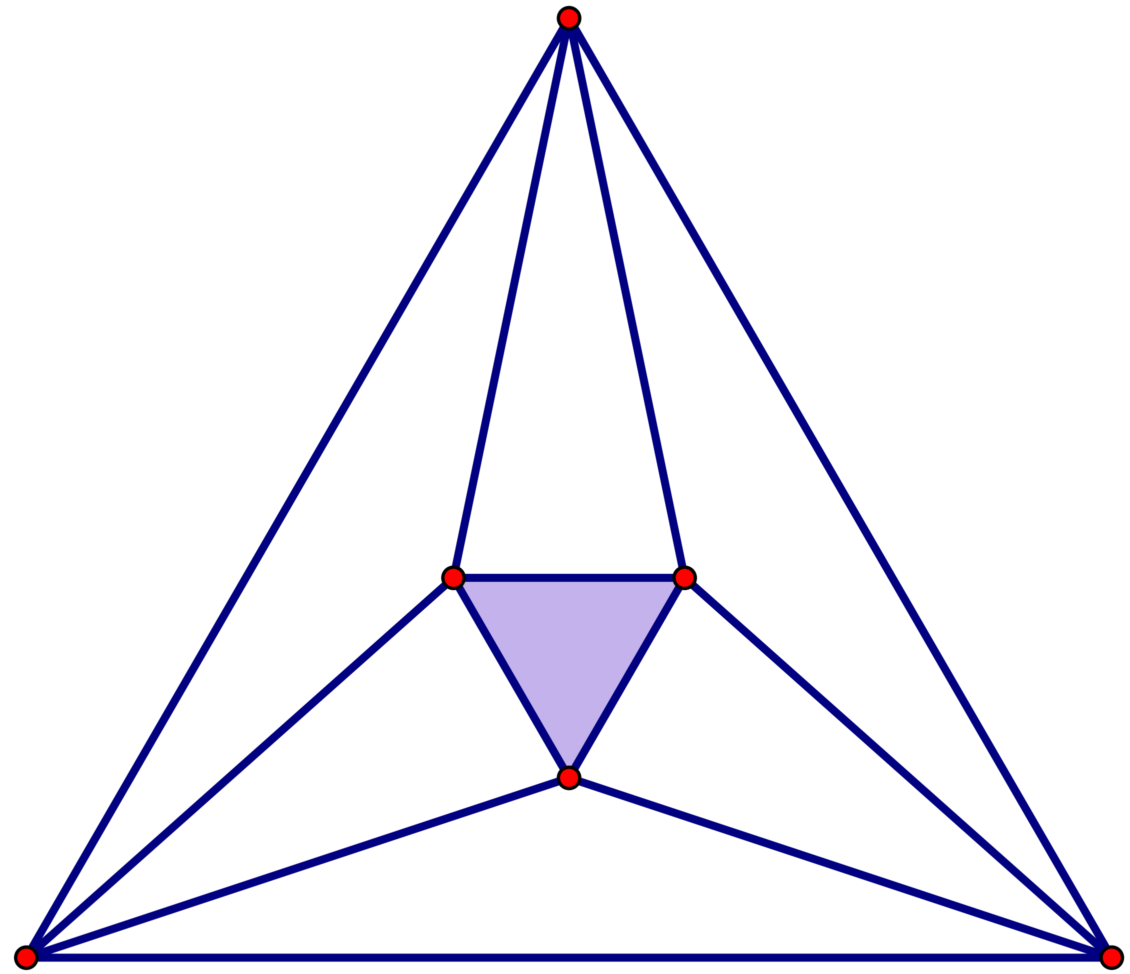


Step 5

- Modify the crosspolytopes:



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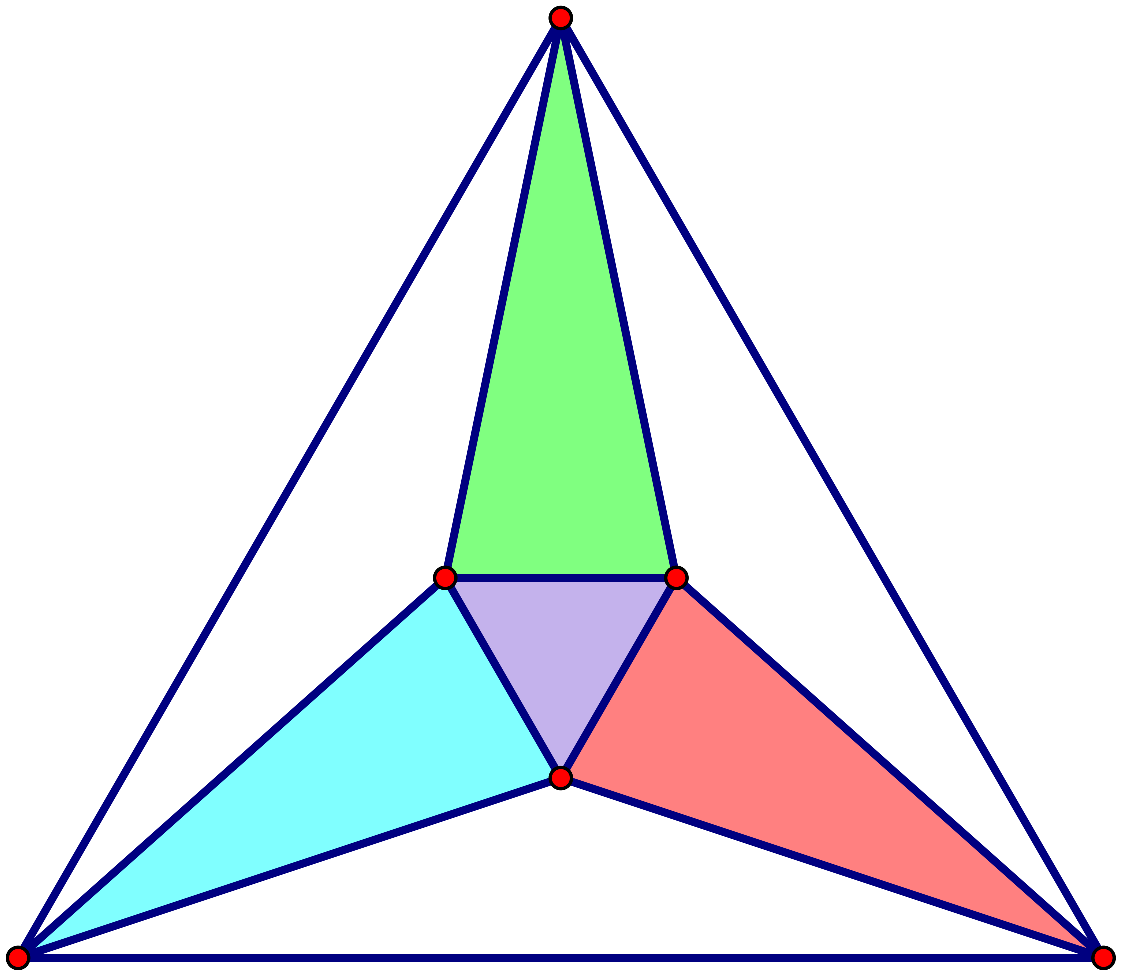

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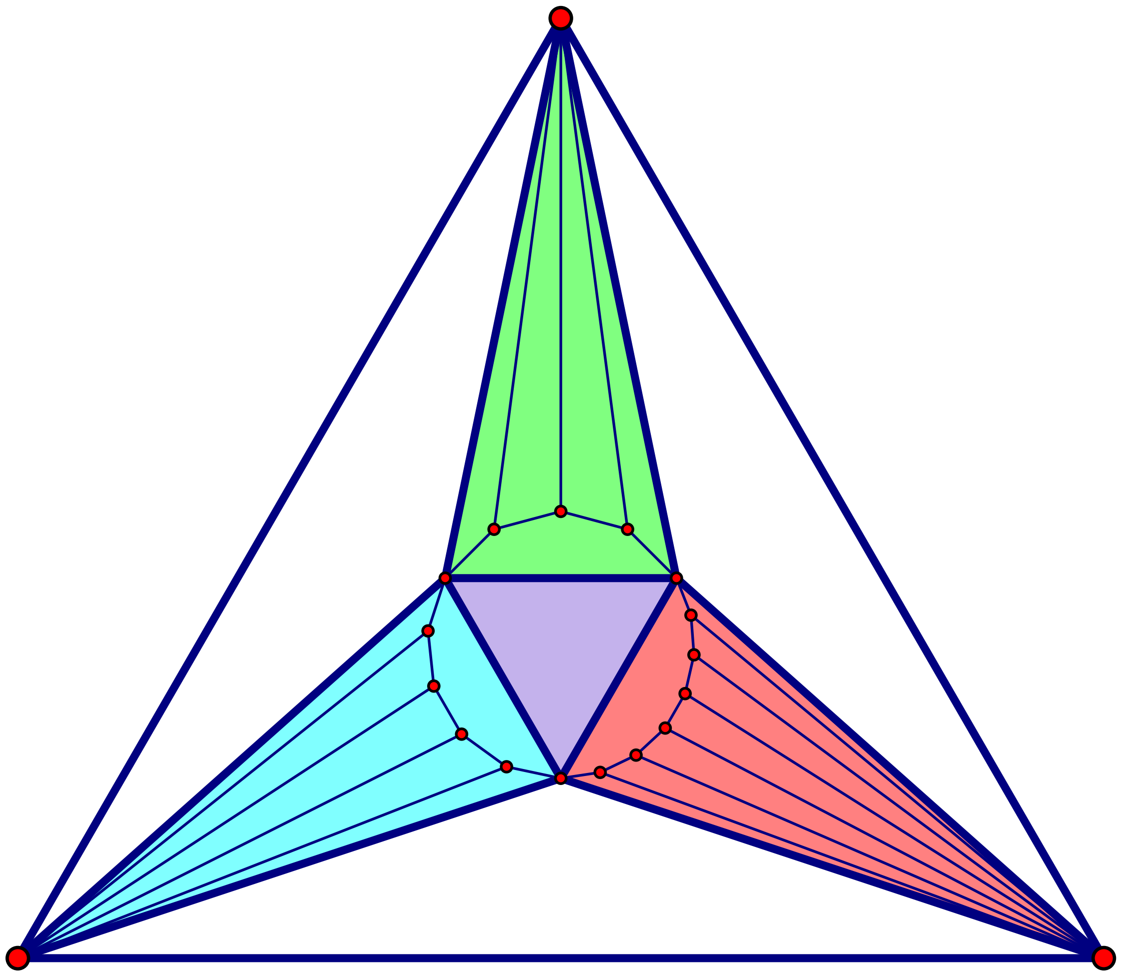
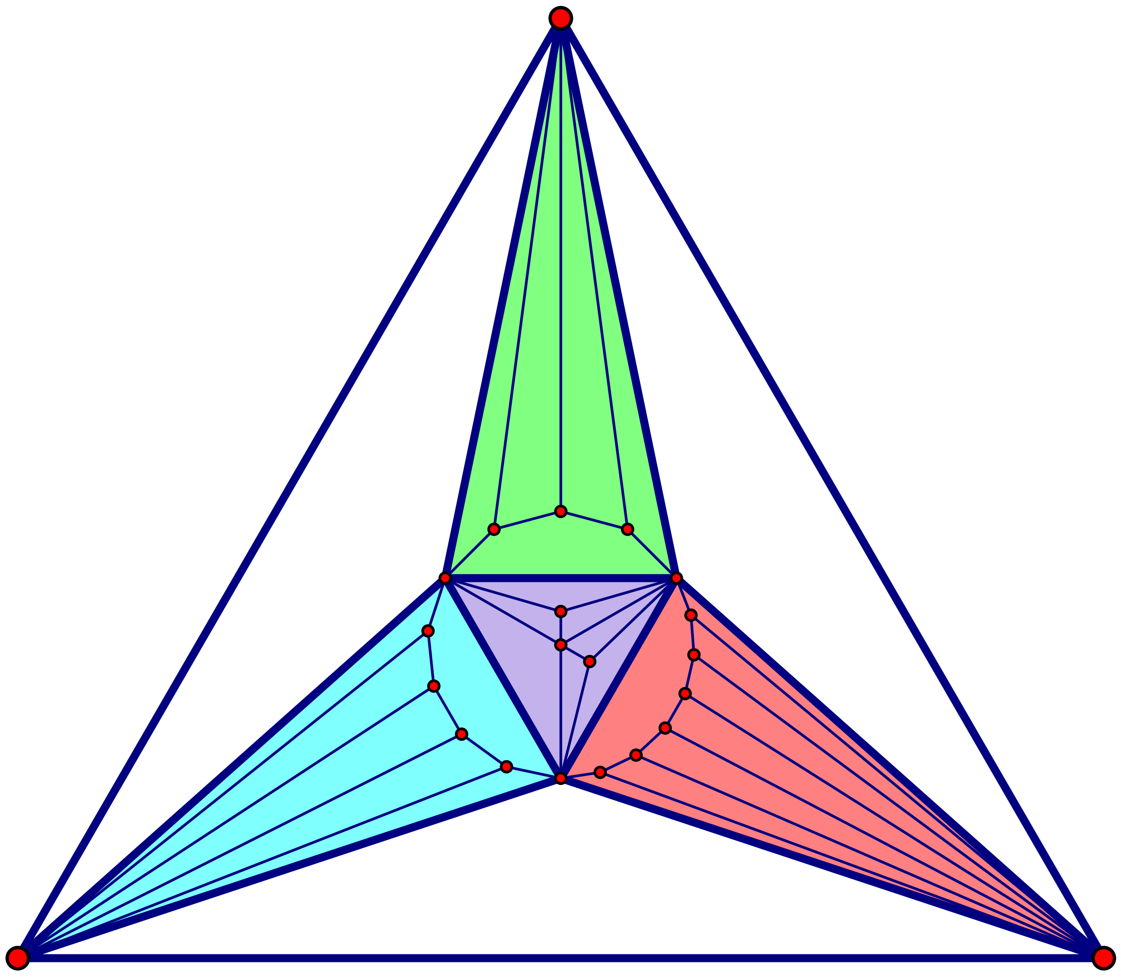
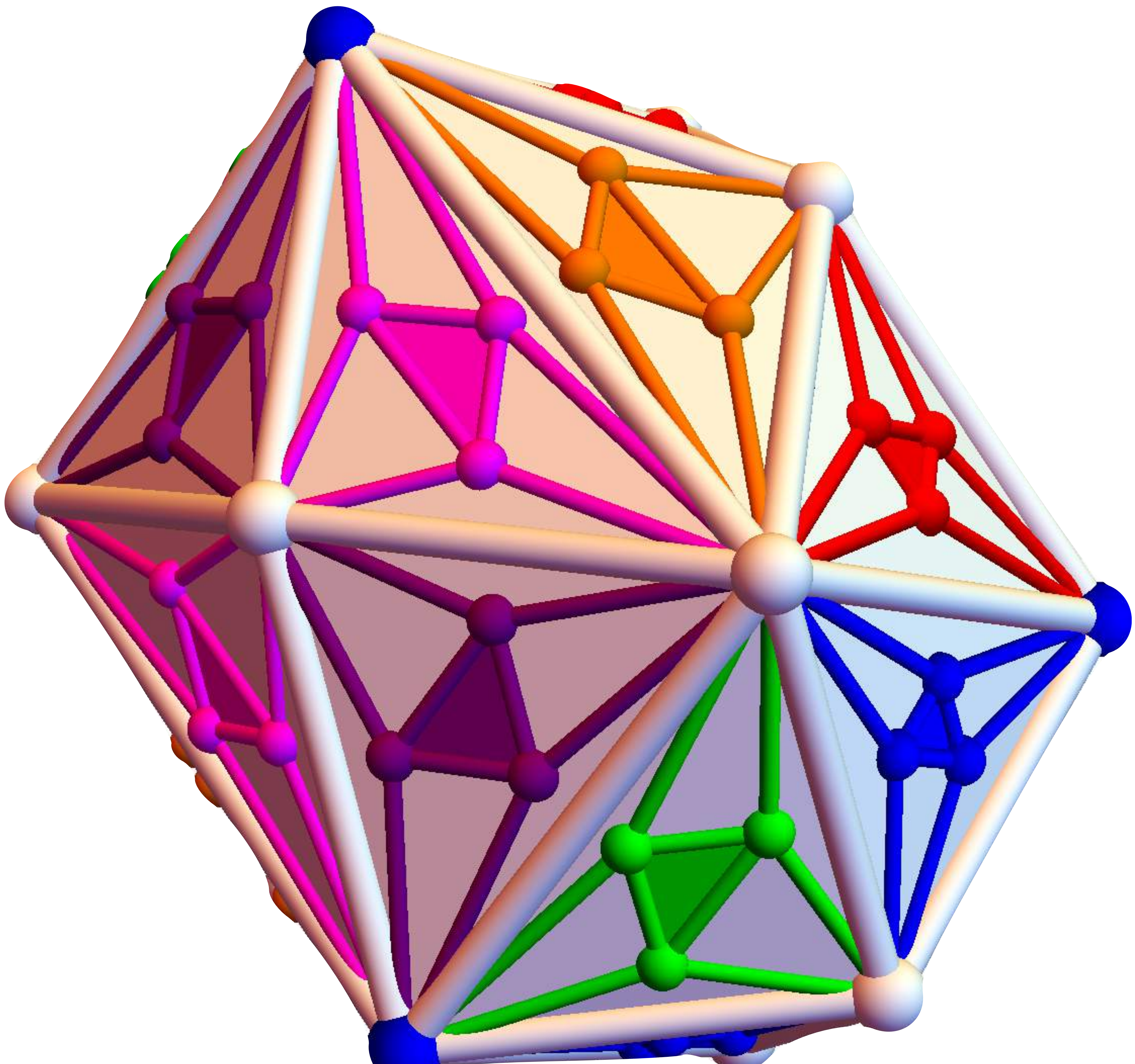
- Modify the crosspolytopes:
  - Paste Schlegel diagrams of polytopes with just one vertex of high degree into the center-adjacent facets.

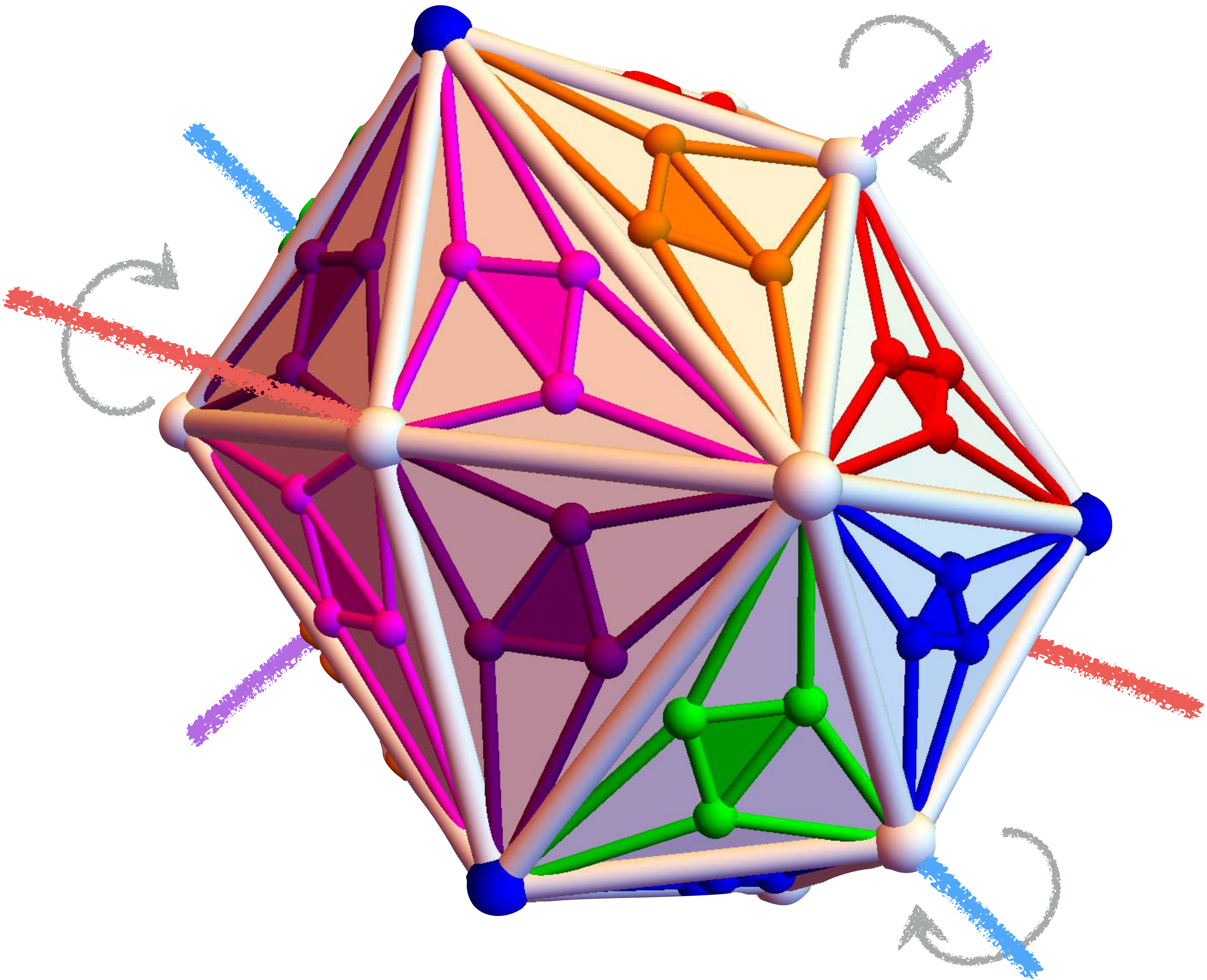


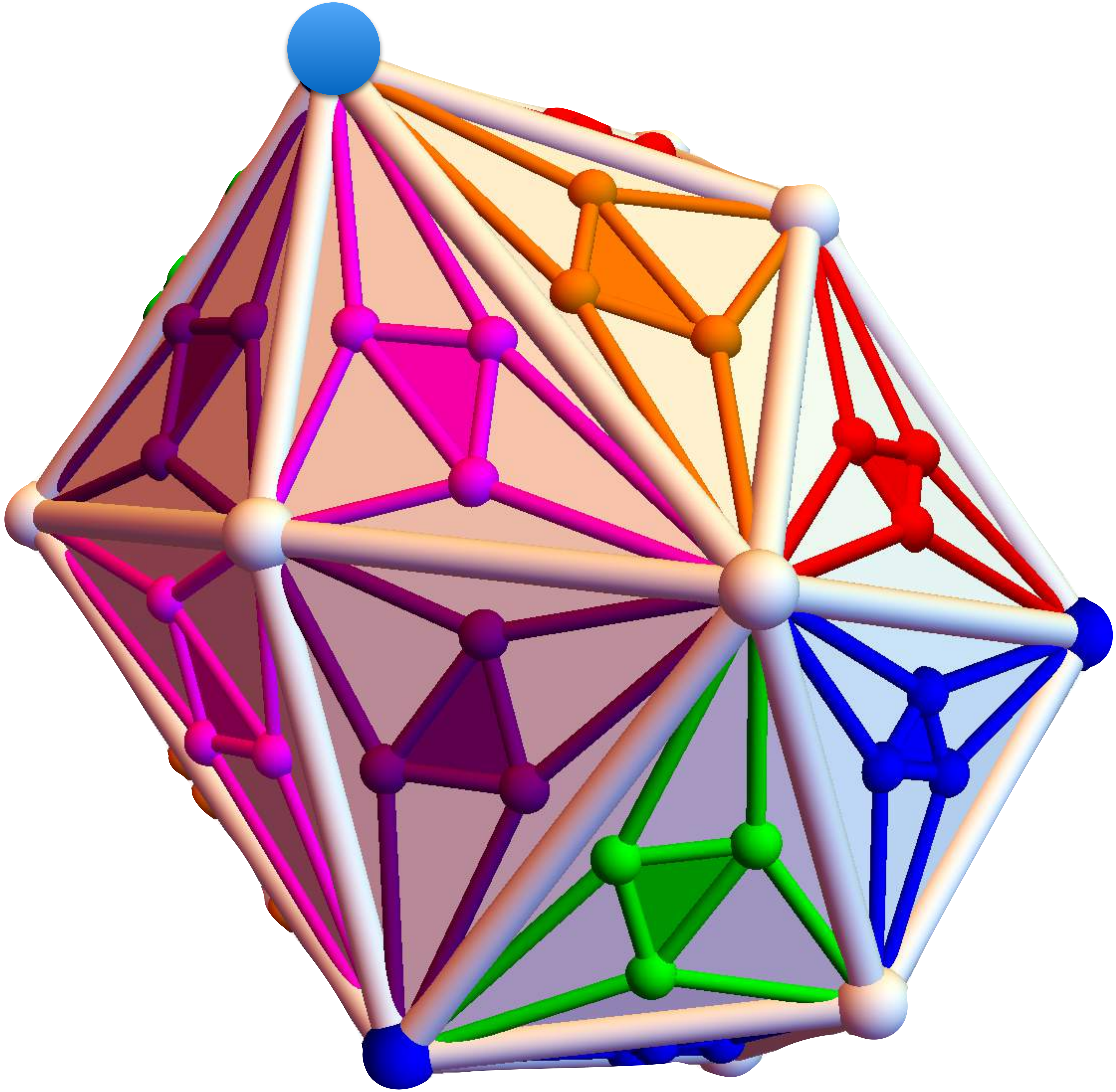
Step 5

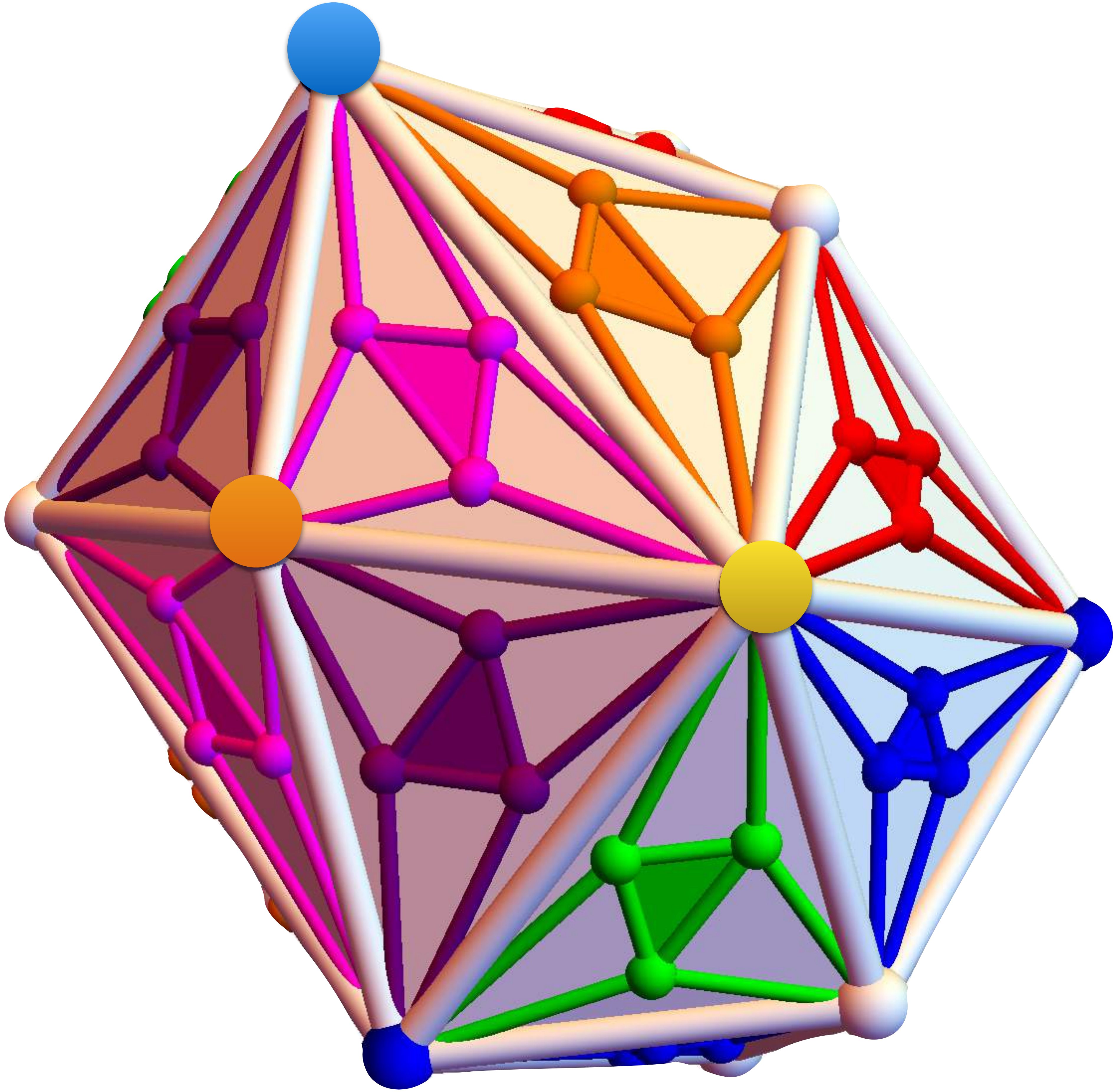
- Modify the crosspolytopes:
  - Paste Schlegel diagrams of polytopes with just one vertex of high degree into the center-adjacent facets.
  - Paste Schlegel diagrams of simplicial polytopes with differing numbers of vertices into the central facet for each automorphism class.



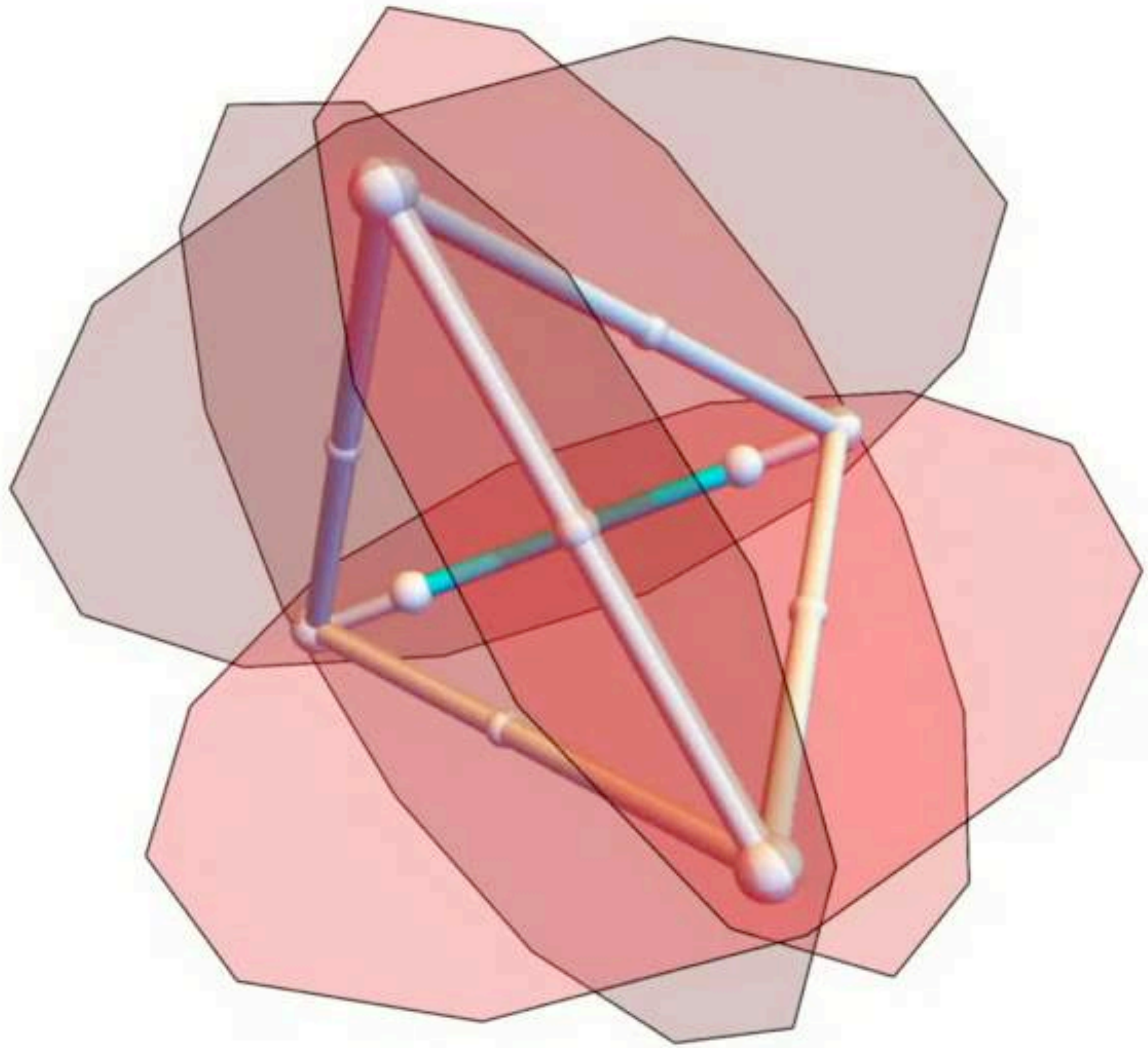


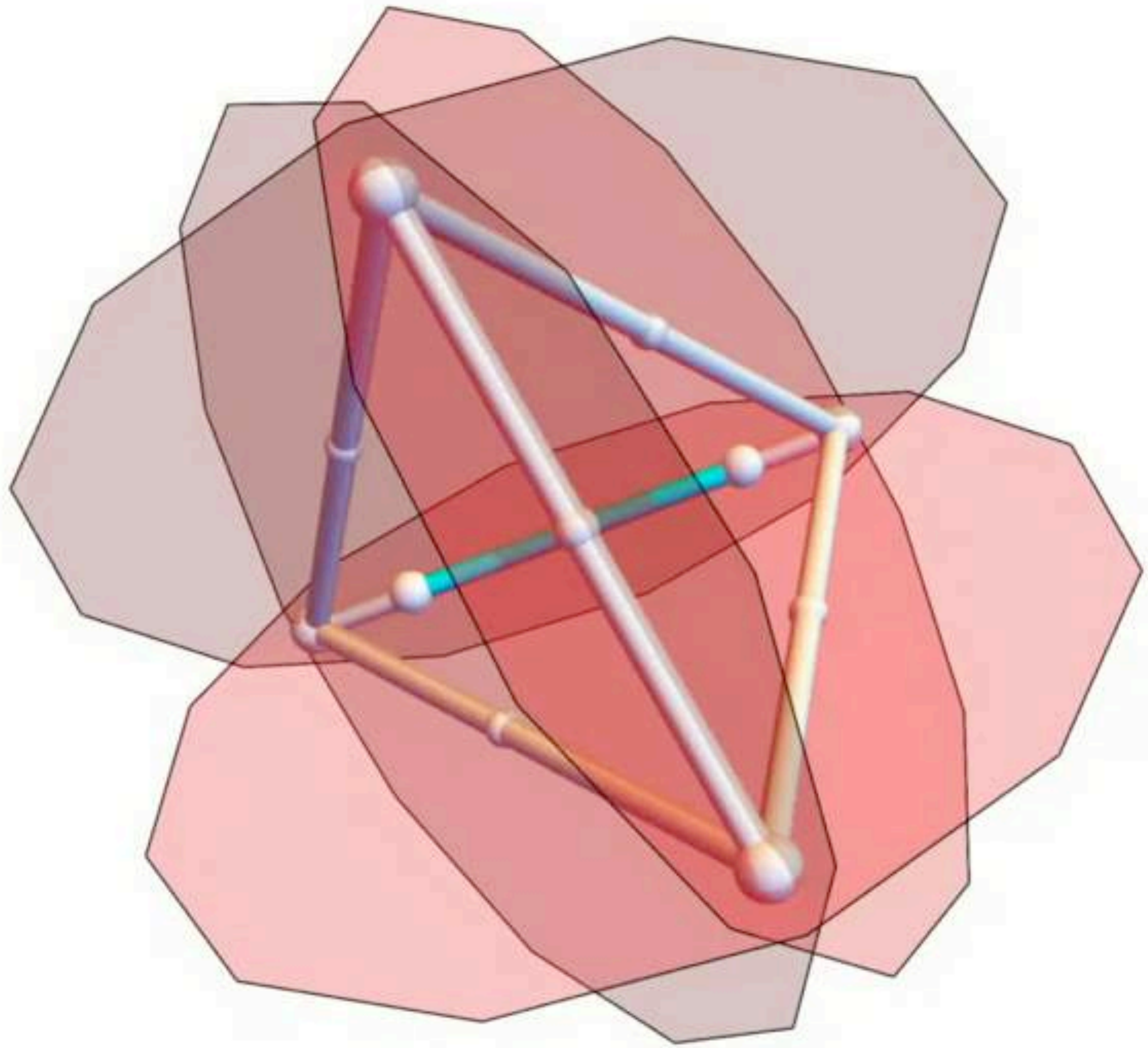


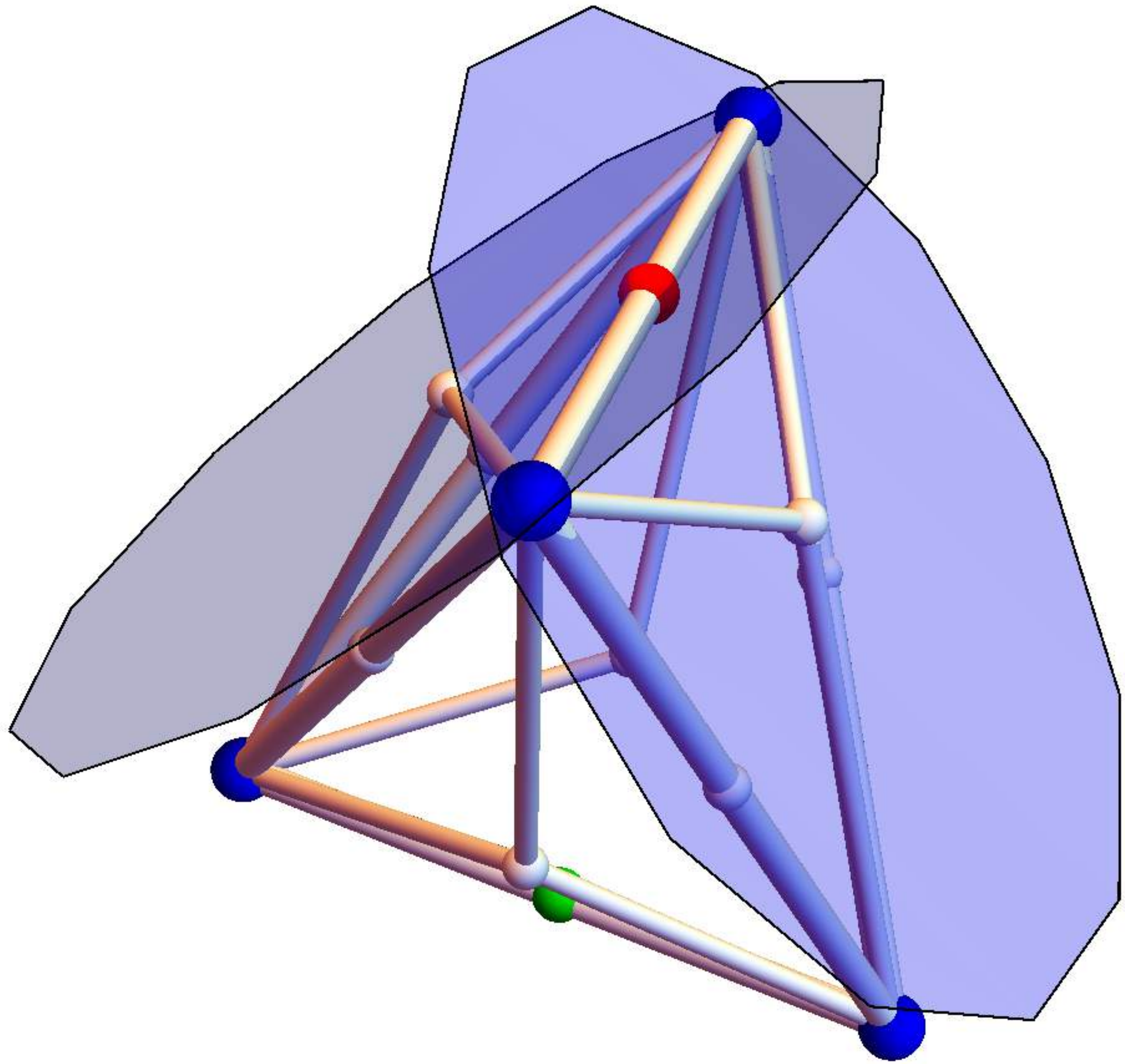


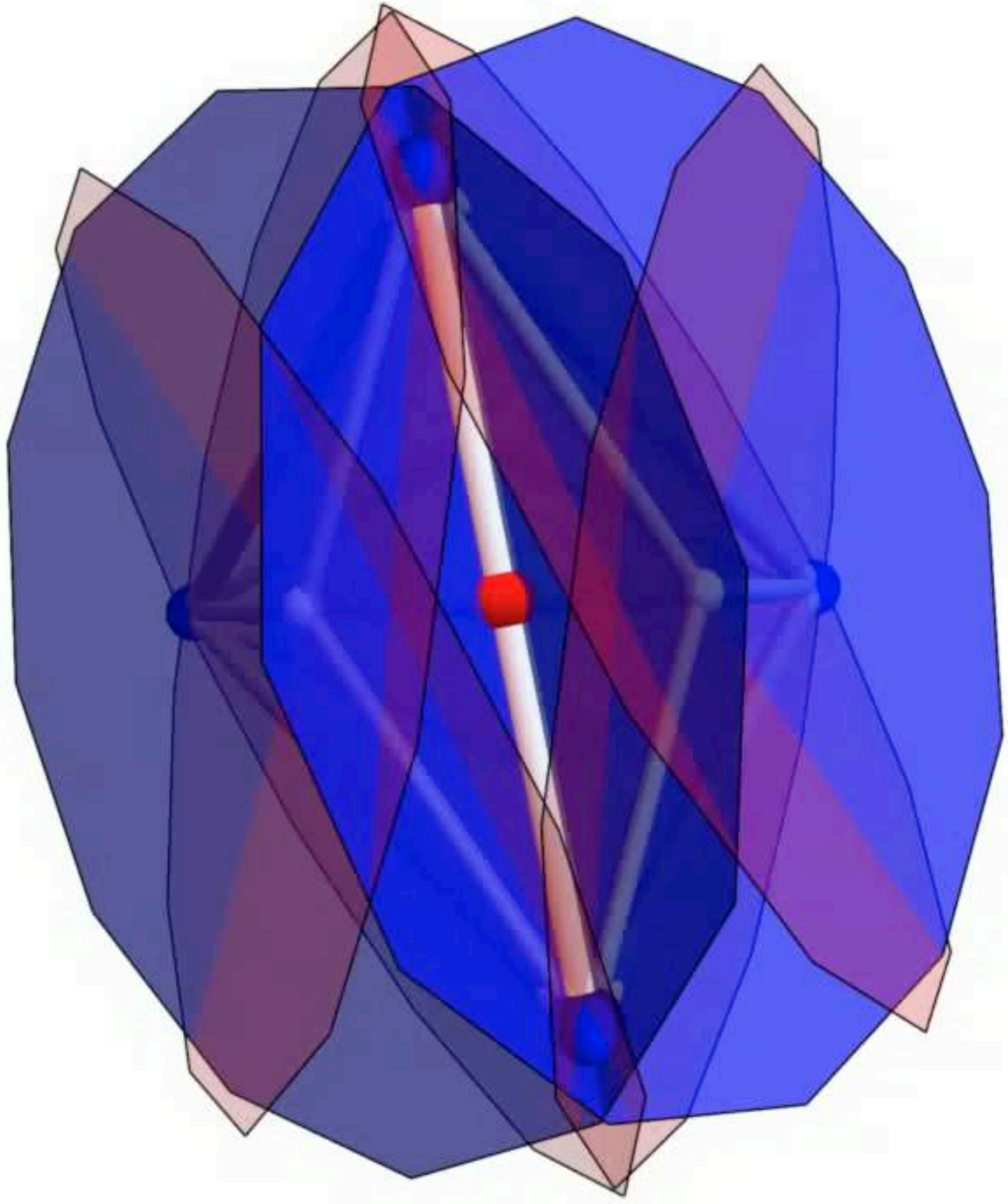


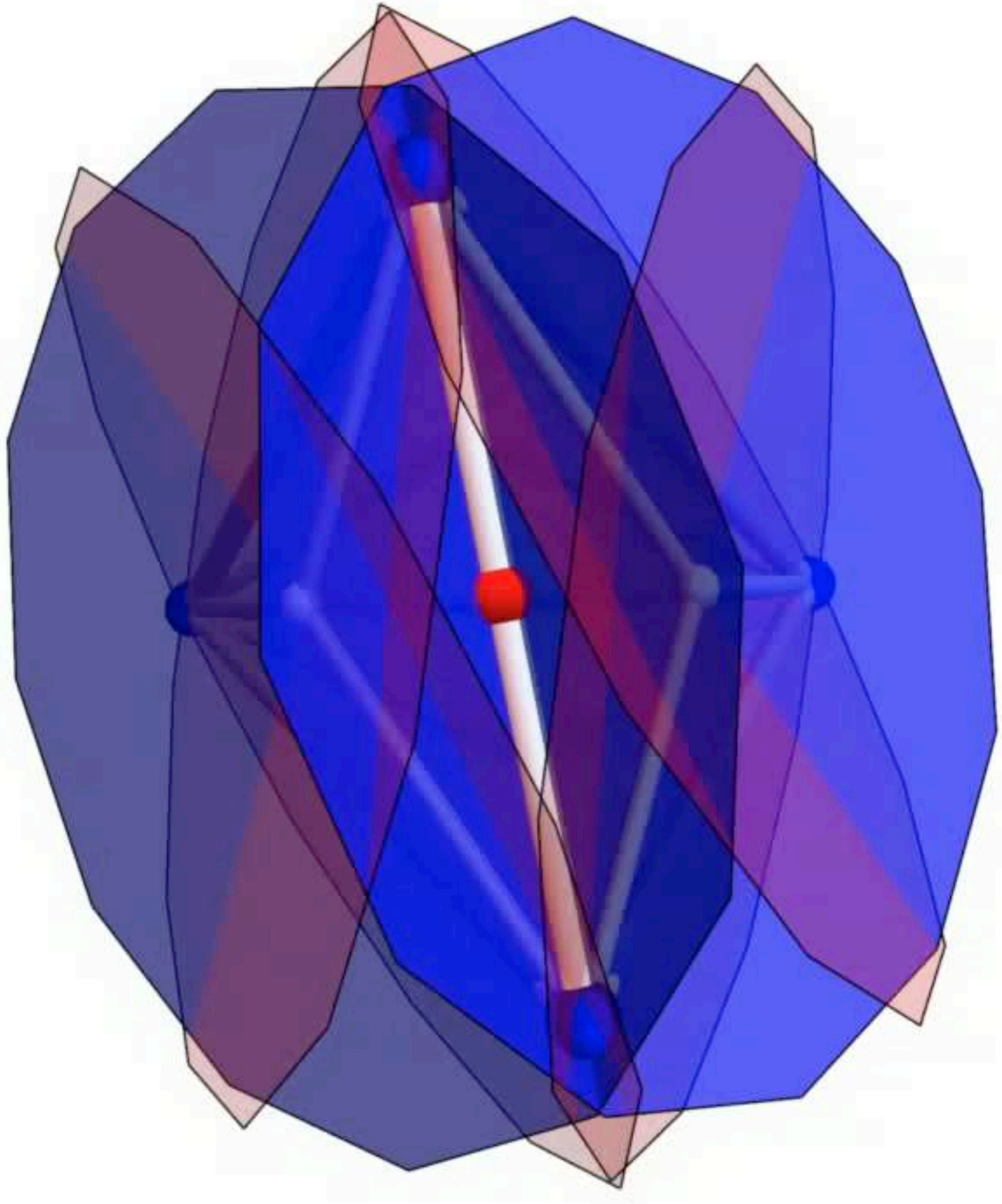
Building the Barycentric Subdivision











Open Questions

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- Is every finite group the automorphism group of a finite locally toroidal abstract 4-polytope?

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- Is every finite group the automorphism group of a finite locally toroidal abstract 4-polytope?
- Can one modify our construction so that the resulting abstract polytope is simplicial or cubical?
- How do algebraic features of the group affect the structural features of the resulting polytope?

- Which countable infinite groups are automorphism groups of locally finite face-to-face tilings of a finite-dimensional real space by topological polytopes?

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    - How about tiling by homeomorphic copies of convex polytopes?

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  - How about tiling by homeomorphic copies of convex polytopes?
  - How about finitely generated infinite groups?

Köszönöm!

Alles Gute zum Geburtstag Egon!
Boldog születésnapot Karoly!

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