A rephrasing of edge colouring by local charts and orientability

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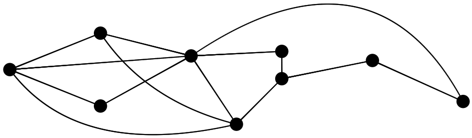
Graphs are the main object of this talk.



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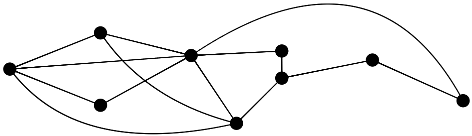
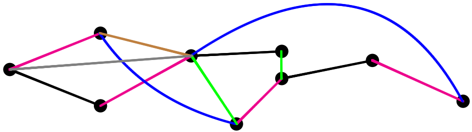
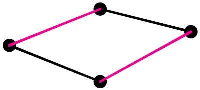


Graphs can be **edge-coloured**:

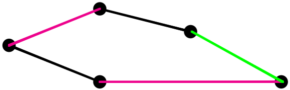


(all distinct colours for any vertex)

Chromatic index (usually q or χ'):
least number of colours needed.

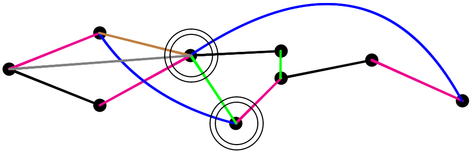


$q = 2$



$q = 3$

Locally, for each vertex we have a **coloured star**.



A coloured star, on a given vertex v, is an **injective map**

$\gamma_v : S_v = \{\text{edges containing } v\} \longrightarrow \{\text{colours}\} \subseteq \mathbf{N}$

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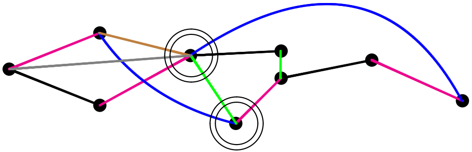

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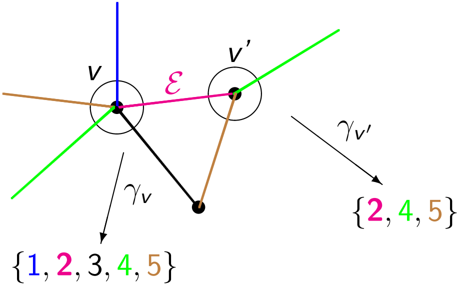
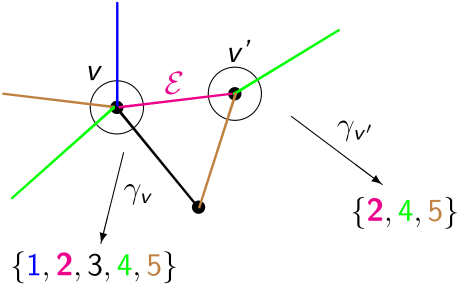
This is somewhat analogous to a **local chart** of a manifold.



γ_v and $\gamma_{v'}$ are **compatible maps**

(they assign **the same colour** to the edge $\mathcal{E}$).



γ_v and $\gamma_{v'}$ are **compatible maps**

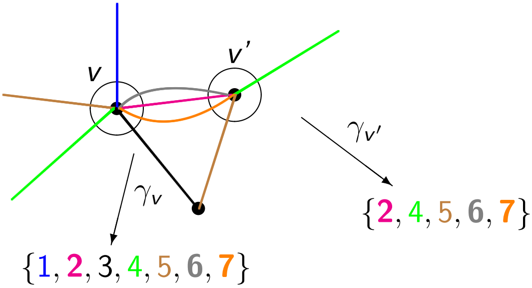
(they assign **the same colour** to the edge $\mathcal{E}$).

The two maps **coincide on the intersection** $S_v \cap S_{v'} = \{\mathcal{E}\}$.

$$\Leftrightarrow \quad \gamma_{v'} \circ \gamma_v^{-1}(\gamma_v(S_v \cap S_{v'})) \longrightarrow \mathbf{N} \quad \text{is the inclusion map}$$

(simply sending **2** to **2**)

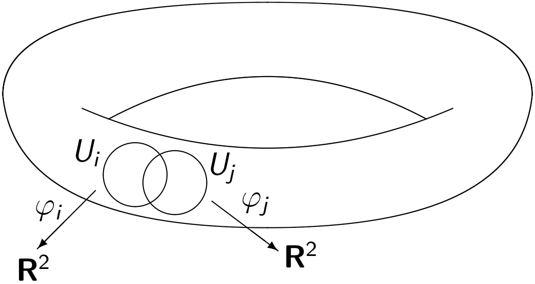
More generally, for a multigraph (multiple edges allowed):



again $\gamma_{v'} \circ \gamma_v^{-1}(\gamma_v(S_v \cap S_{v'})) \longrightarrow \mathbf{N}$ is the inclusion map .

(now the domain is $\{2, 6, 7\}$)

The analogy with manifolds is now easy to draw.



φ_i, φ_j are homeomorphisms (local charts).

$$\varphi_j \circ \varphi_i^{-1}(\varphi_i(U_i \cap U_j)) \longrightarrow \mathbf{R^2} \quad \text{is a } \textbf{differentiable} \text{ map}$$

$$\left[\left[\quad \gamma_v \circ \gamma_u^{-1}(\gamma_u(S_u \cap S_v)) \longrightarrow \mathbf{N} \quad \text{is the } \textbf{inclusion} \text{ map} \quad \right]\right]$$

Let G be a graph of *degree** Δ.
(* largest number of edges containing a vertex, over all vertices)

Vizing's Theorem:
Least number of colours for an edge colouring $\in \{\Delta, \Delta + 1\}$.

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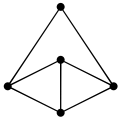
Least number of colours for an edge colouring $\in \{\Delta, \Delta + 1\}$.
So we can give the following

Definition

Let v be a vertex of G. A **neighborhood** of v is the set S_v of all edges containing v. A **local chart** on v is an injective map $\gamma_v : S_v \to \{1, 2, ..., \Delta + 1\}$. An **atlas** on G is a set of local charts $\{\gamma_v\}_{v \in V}$ such that $\gamma_{v'} \circ \gamma_v^{-1} : \gamma_v(S_v \cap S_{v'}) \to \{1, 2, ..., \Delta + 1\}$ is the inclusion map, for any adjacent vertices v, v'.

(similarly, for multigraphs – generalised Vizing's Theorem...)

For example, this graph has $\Delta = 3$ and requires 4 colours.



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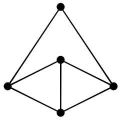
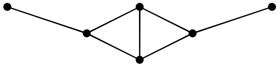


Proof: untie the two upper edges.



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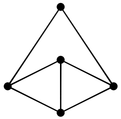

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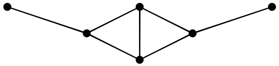
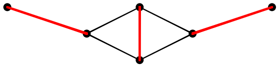
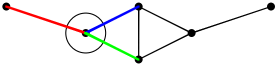


Now try to use only 3 colours...



$\Rightarrow$ colour **red** is necessary at both ends.

Using the language of manifolds...
we are **extending** an atlas, starting from a local chart:



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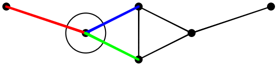

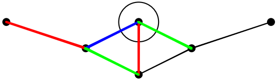

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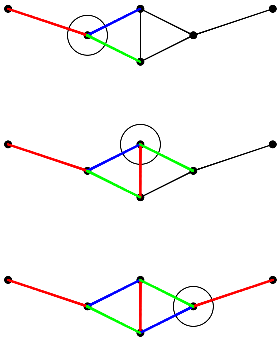

STOP: in the end we cannot eventually identify the extremes.

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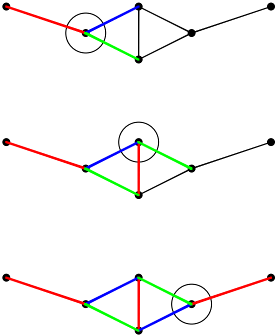

STOP: in the end we cannot eventually identify the extremes.
This sounds like a **well known phenomenon**, for manifolds!

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Take a disk ○ whose other side is ●.

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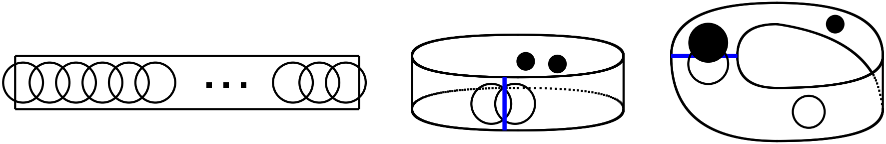
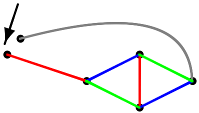
Take a disk ○ whose other side is ●.

Cover a strip, then **identify the extremal edges** of the strip.



The second identification is not allowed
(if are looking for a **global** orientation)

"Non-orientable" $\longleftrightarrow$ **4 colours needed** (as above)



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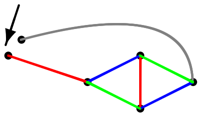
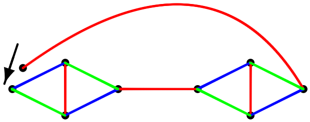


"Orientable": $\longleftrightarrow$ **3 colours suffice**.



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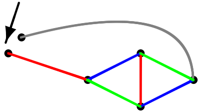

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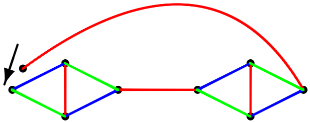

Vizing's Theorem $\Rightarrow$	CLASS 1	CLASS 2
	Δ colours	$\Delta + 1$ colours
"ORIENTABILITY"......	**YES**	**NO**

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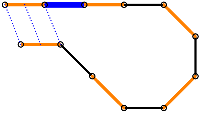
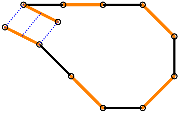
Degree 2: even more easy to see than with degree 3.

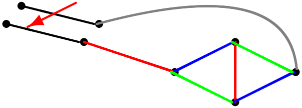


Non-orientable
(odd cycle, 3 colours)

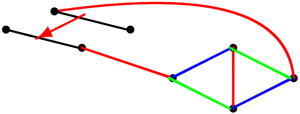


Orientable
(even cycle, 2 colours)

In the degree-3 case, imagine to append two extremal edges. Orientability depends on the way we identify them.



4 colours needed



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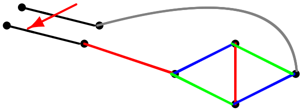

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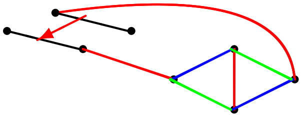

3 colours suffice

Again a clear resemblance with $\boxed{\text{the Möbius strip}}$!

Orientability gives a new way of looking at **critical graphs**.

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- it requires $\Delta + 1$ colours;

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A graph of degree Δ is (edge)-**critical** if:

- it requires $\Delta + 1$ colours;
- after removing any edge, the required colours are Δ.

(so it passes from class 2 to class 1 whenever we remove an edge)
($\Leftrightarrow$ it becomes orientable whenever we remove an edge)

Critical graphs still provide many challenging problems.

22 vertices are the least for a **3-critical** graph of even order (Goldberg 1981, Brinkmann-Steffen 1997).

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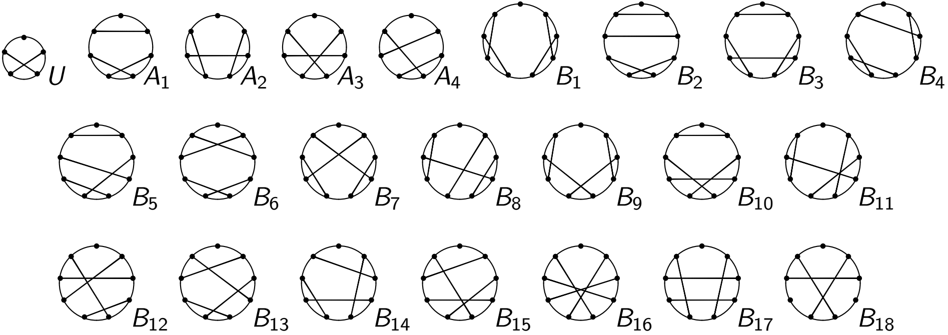
Constructions of critical graphs often require a computer aid.

I.T. Jacobsen, *On critical graphs with chromatic index* 4, Discr. Math. **9** (1974), pp. 265-276.

Classification of 3-critical graphs with 5, 7, 9 vertices.

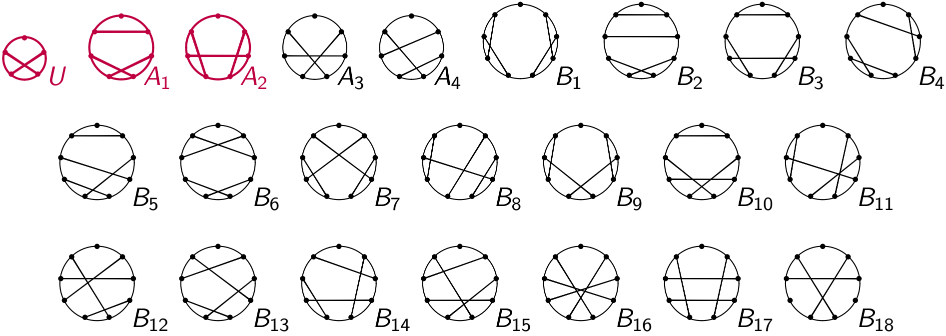


Using the language of orientable atlases, we get

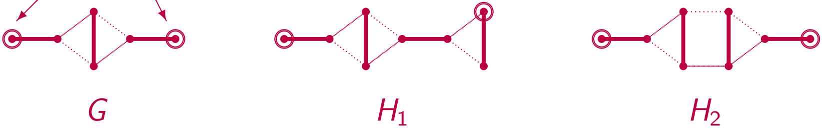
$\boxed{\textbf{an alternative classification.}}$

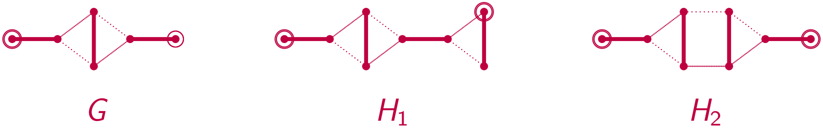
Apparently "distant" graphs become now of the same type.

Just a few **patterns** describe almost all graphs.



identification gives U

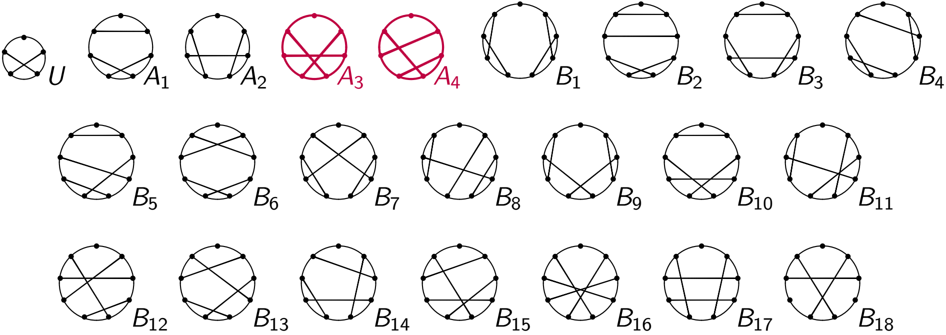




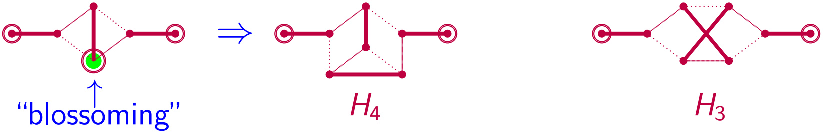
Key concept:

Transmission of colours from one extreme to the other.

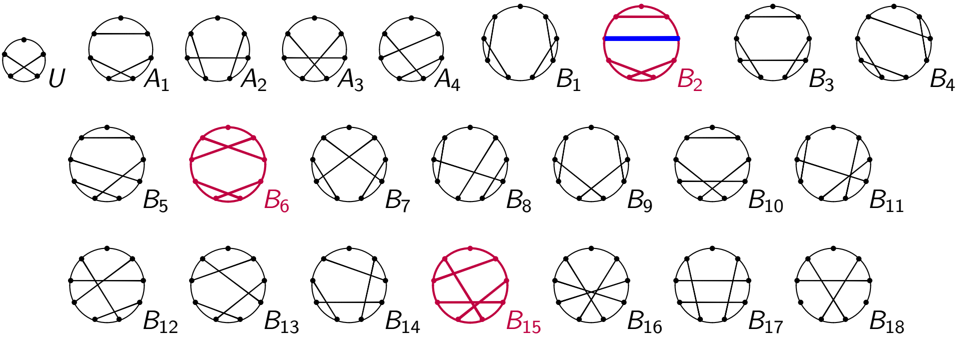
Extremal edges need the same colour $\Rightarrow$ **LOSS** of orientability after the identification.



U
A_1
A_2
A_3
A_4
B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8
B_9
B_{10}
B_{11}
B_{12}
B_{13}
B_{14}
B_{15}
B_{16}
B_{17}
B_{18}



$\Rightarrow$
"blossoming"
H_4
H_3


U
A1
A2
A3
A4
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
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B12
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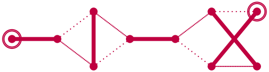
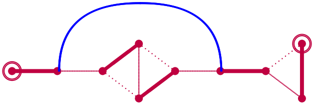
B15
B16
B17
B18

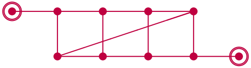


J_6

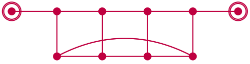
J_{15}



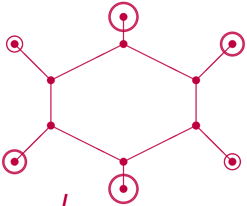
J_2



J_{16}

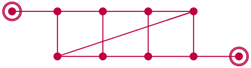


J_{17}



J_{18}

Transmission can be recognised (less easily) also in B_{16} and B_{17}.



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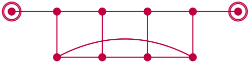
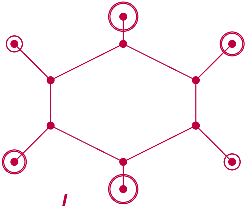


J_{17}



J_{18}

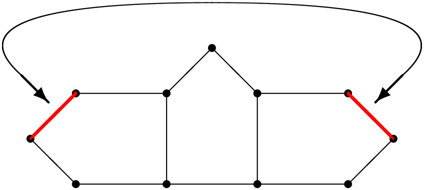
Transmission can be recognised (less easily) also in B_{16} and B_{17}.

The last graph, J_{18}, is an exception.

It is not that comfortable to describe a transmission.

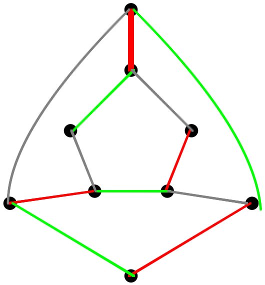
B_{18} has only 12 edges!
(Criticality is more "structural" than in the other cases.)

Nonetheless, also B_{18} has the “Möbius strip” syndrome:



To obtain B_{18}: **twist** and then identify the two edges.

If we do not twist, we have an **orientable** graph:



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- ⋄ Constructing families of 3-critical graphs using polygons;
- ⋄ Using the language of atlases in a more effective way;
- ⋄ What happens with k-critical graphs, $k \geq 4$?

The above ideas have been collected in a recent paper (see the journal Graphs and Combinatorics).