On contact numbers of totally separable unit sphere packings

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Many thanks to my teachers:

Károly Bezdek

and

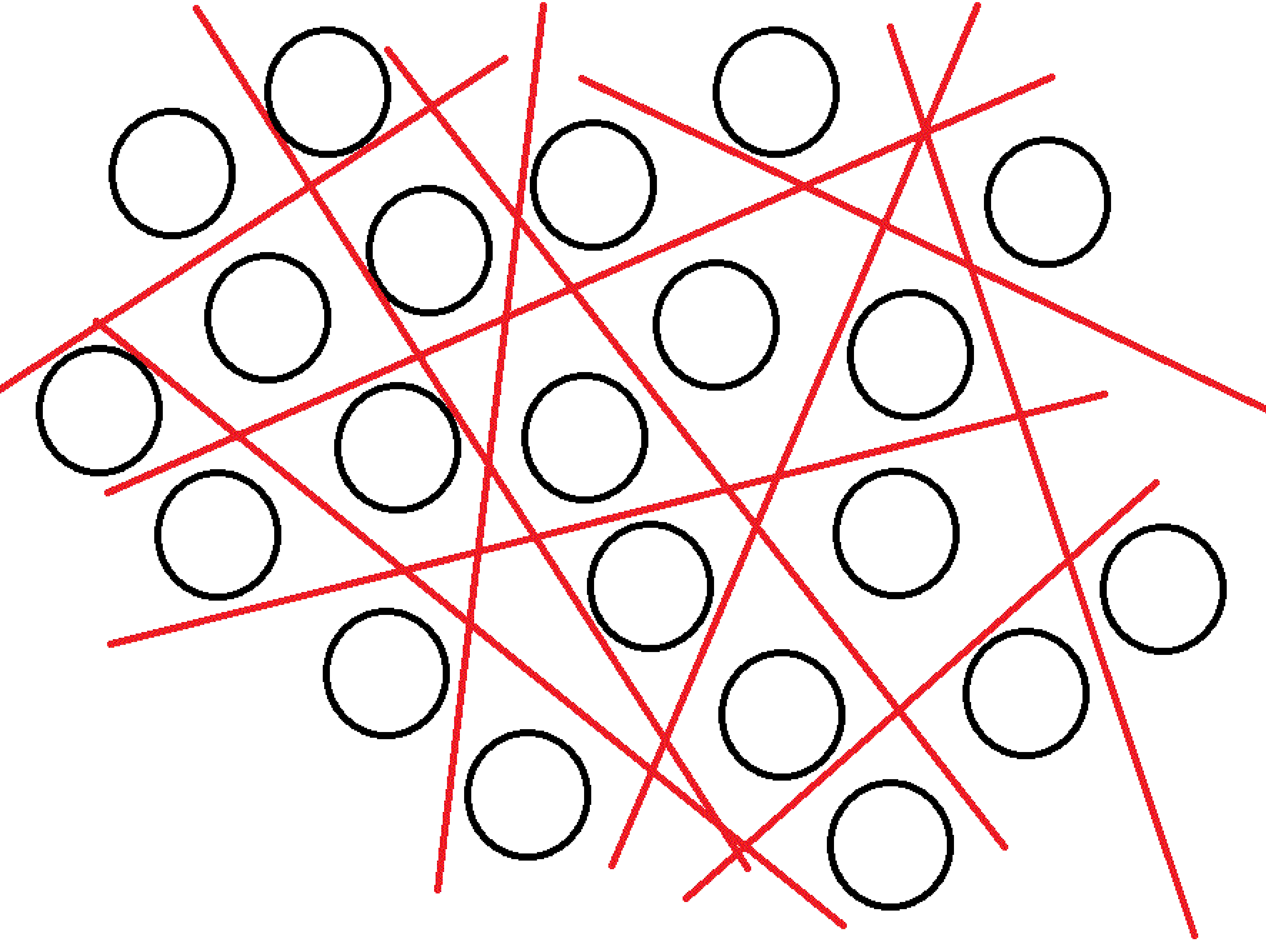
Károly Böröczky

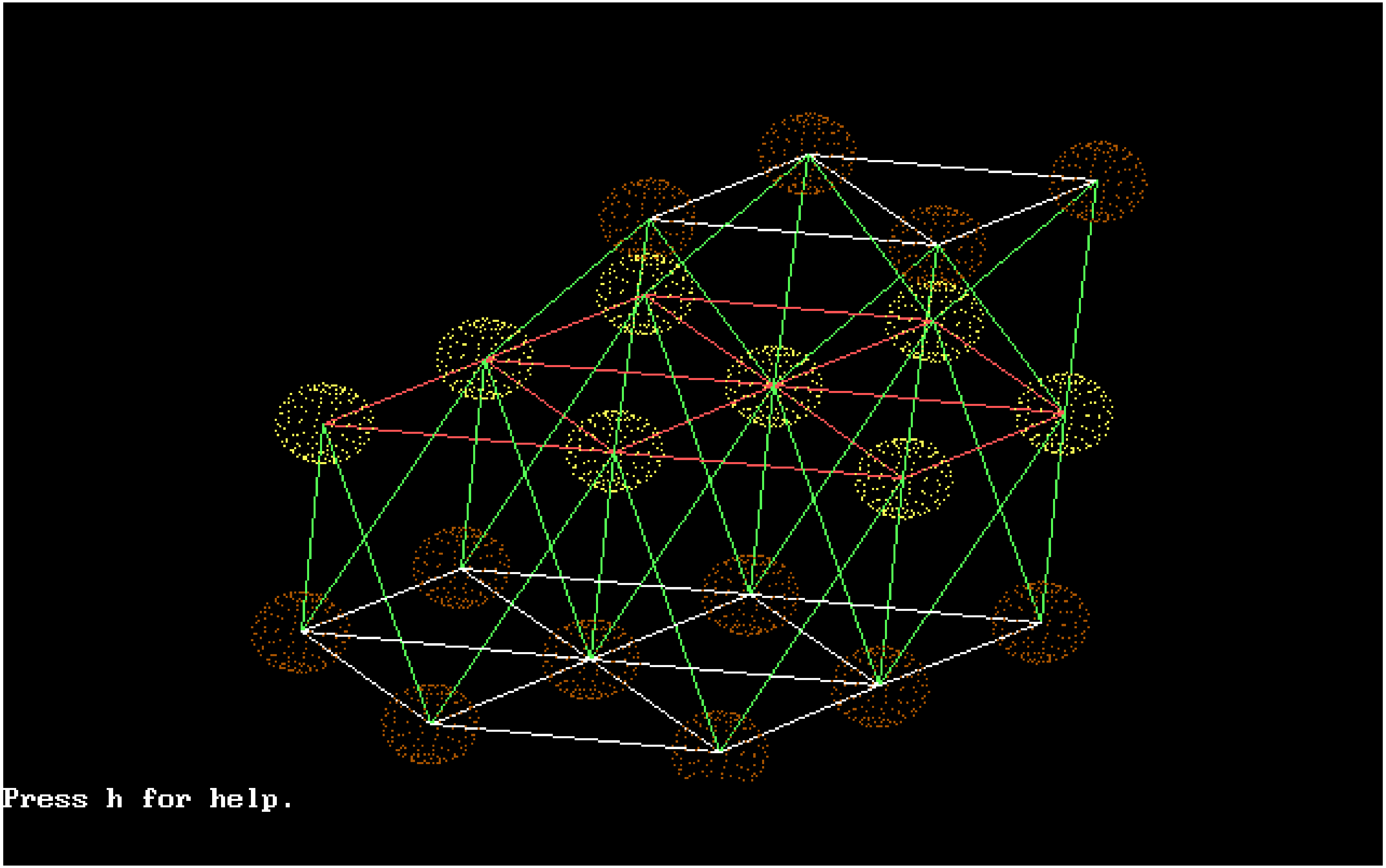

scree
Gefährliches Geröll
scree
Eine unvorsichtige Bewegung . . . ein falscher Griff . . . und die Falle schnappt zu.
PARKER
Weltbekannt durch MONOPOLY

Definition (G. Fejes Tóth, L. Fejes Tóth, 1973) :
A packing of balls is **totally separable** *if* any two *balls can be separated by a* hyperplane *of* E^d, *disjoint from the interior of* each *unit ball in the packing.*

Special case: *The centers of the balls are from* ***a*** *grid / lattice* .

More special case: *The centers are from* ***the*** *lattice* Z^d .

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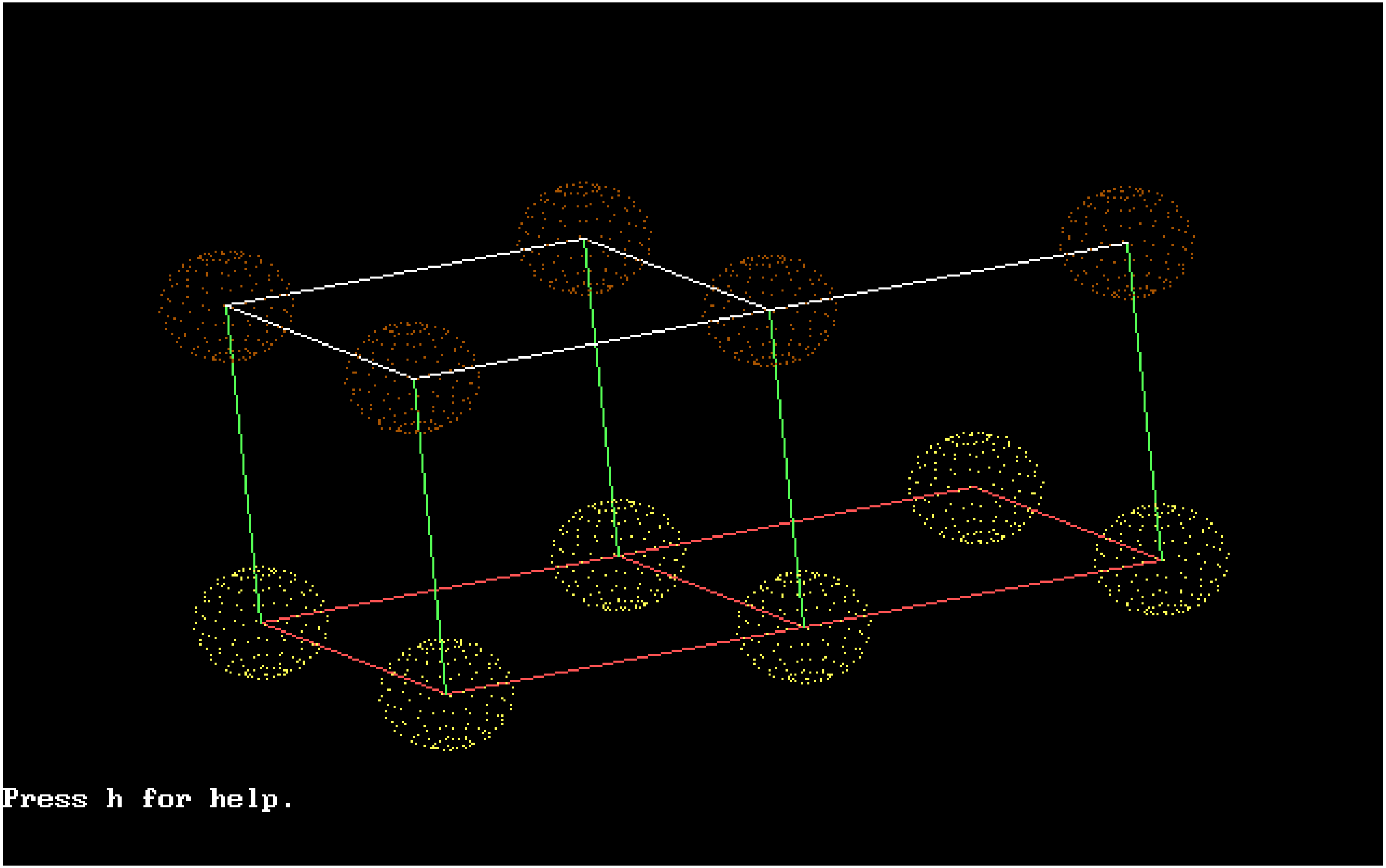
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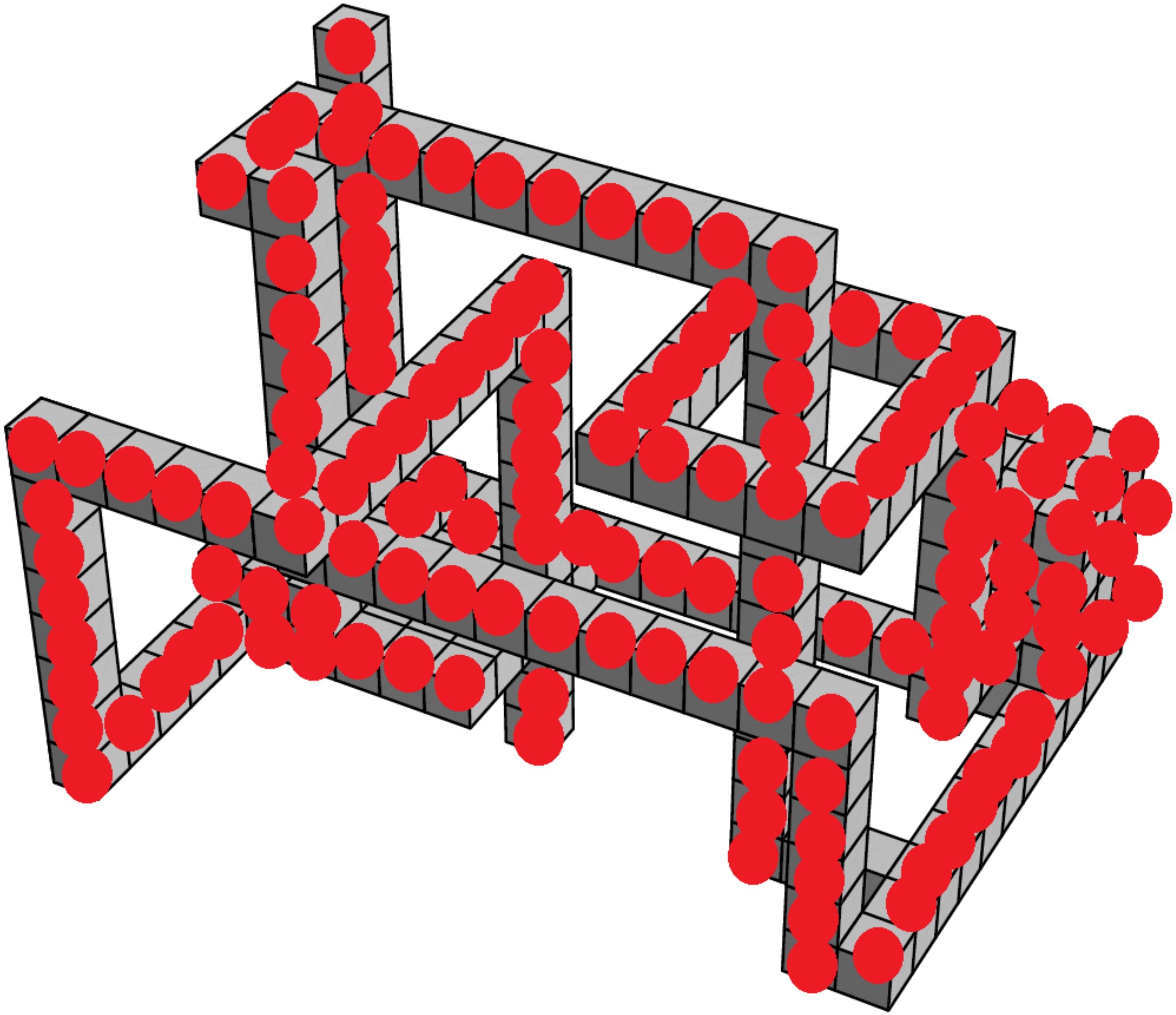
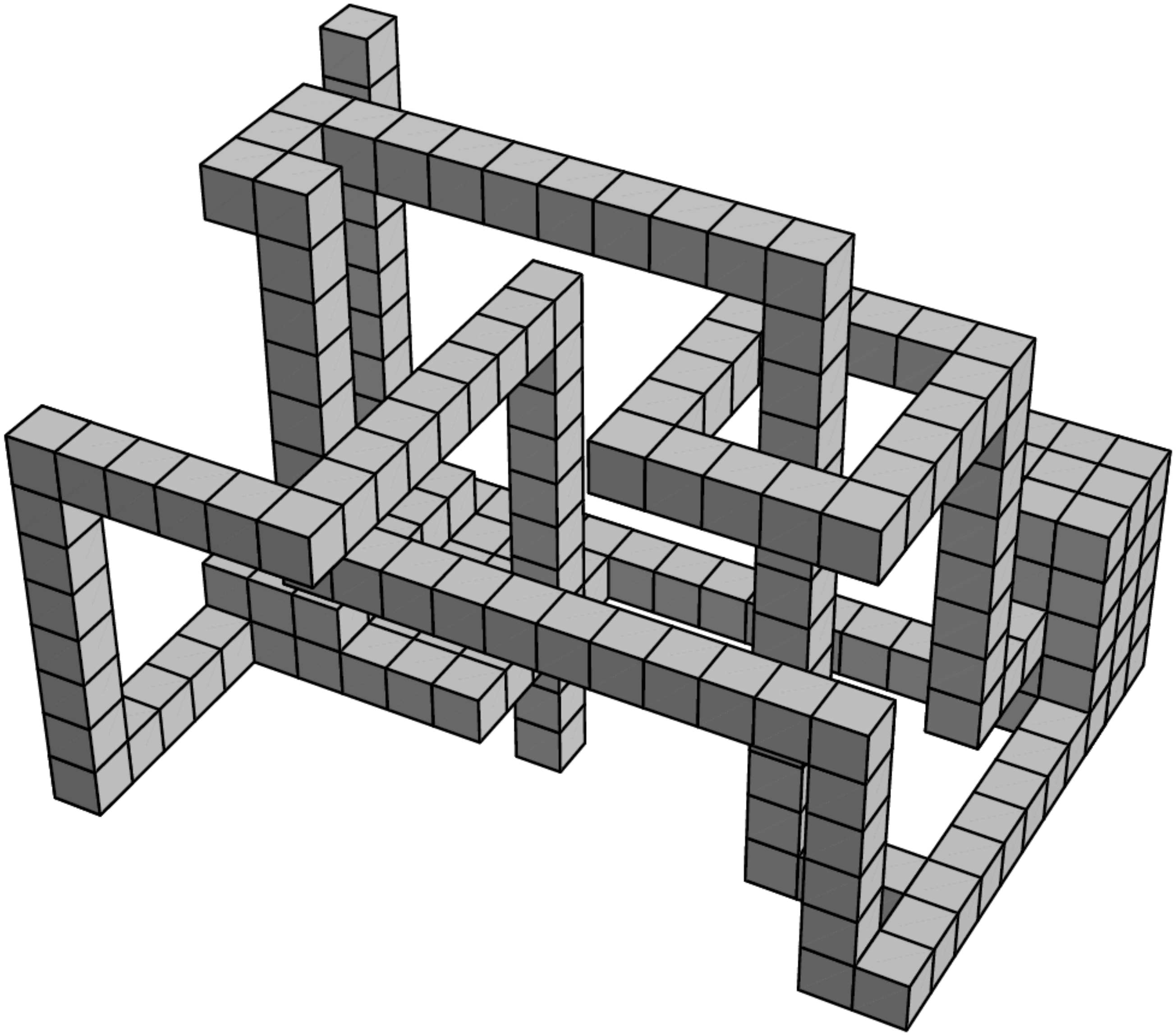
More special case: *The centers are from* ***the*** *lattice* Z^d

$\Rightarrow$ balls can be replaced by **cubes** $\Rightarrow$ **polyominoes**

or **box-polytopes**

$\Rightarrow$ **maximal touching nu. ≡ minimal surface**





τ_d = *kissing number* = max. number of balls, touching **one** fixed ball,

$c_d(n)$ = *contact number* = max. touching number of n balls in E^d ,

c(*n,d*) = *contact number for* ***<u>totally separable</u> packings***,

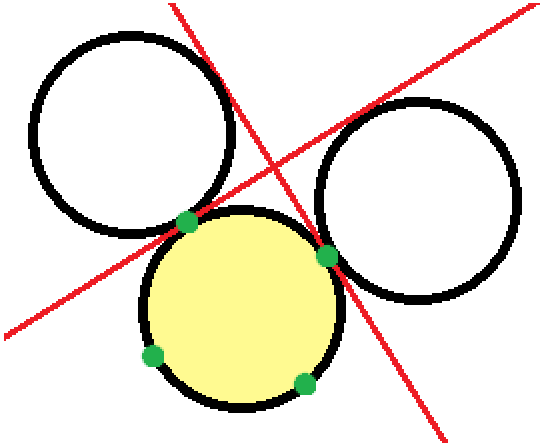
$c_Z(n,d)$ = *contact number for* $\mathbf{Z^d}$ (balls=cubes, polyomino).

Clearly:

$$c_Z(n,d) \leq c\,(n,d) \leq c_d(n) \leq \tfrac{1}{2}\tau_d n$$

$$c\,(n,d) \leq dn$$

Statement: $\mathbf{c}\,(\boldsymbol{n},\boldsymbol{d}) \le \boldsymbol{dn}$ (totally separable)



"kissing points" $\mathbf{t_1}$, ... , $\mathbf{t}_{\boldsymbol{k}}$

totally separable => spherical distance$(t_{\mathbf{i}},t_{\mathbf{j}}) \ge {}^{\pi}/_{2}$

=> $k \le 2d$

(equality iff $\mathbf{t_1}$, ... , $\mathbf{t}_{\boldsymbol{k}}$ form a regular inscribed cross-polytope)

=> "Handshaking Thm."

=> $c\,(n,d) \le dn$.

Preliminaries:

Harborth /1974/: $c_2(n) = [\ 3n - \sqrt{12n-3}\ ]$

Bezdek, Reid /2013/: $c_3(n) \leq 6n - 0.926n^{\frac{2}{3}}$

Bezdek /2002/, $d \geq 4$: $c_d(n) \leq \frac{1}{2}\tau_d\, n - \frac{1}{2^d}\delta_d^{-\frac{d-1}{d}}\, n^{\frac{d-1}{d}}$

Bezdek /2014/: $c(n,2) = [\ 2n - 2\sqrt{n}\ ]$

well known: $c_Z(n,2) = [\ 2n - 2\sqrt{n}\ ]$

Alonso, Cerf /1996/: $c_Z(n,3) = 3n - 3n^{\frac{2}{3}} + o(n^{\frac{2}{3}})$

open: $c(n,d) = c_Z(n,d)$ **?** for $n>2$.

where $\boldsymbol{\delta}_d$ *denotes the largest possible density for* (infinite) *packings of unit balls in* E^d .

New results:

Thm 1: $c_{\mathbb{Z}}(n,d) \le \left\lfloor dn - dn^{\frac{d-1}{d}} \right\rfloor$ ($2 \le n,d$)

(sharp for: d=2, $\forall n$, d=3, $n=k^d$ = large cube, not sharp: d=3, n=5) .

Thm 2: $c(n,d) \le \left\lfloor dn - \frac{n^{\frac{d-1}{d}}}{2d^{\frac{d-1}{2}}} \right\rfloor$ $n > 1$, $d \ge 4$

Thm 3: $c(n,3) < \left\lfloor 3n - 1.346\, n^{\frac{2}{3}} \right\rfloor$

http://arxiv.org/abs/1501.07907

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<u>Proof</u>s of Thm 1: $c_{\mathbb{Z}}(n,d) \le \lfloor dn - dn^{\frac{d-1}{d}} \rfloor$

I) /elementary/: Z^d contains of *levels* $\Rightarrow$ **induction on** *d* :

$$c_{\mathbb{Z}}(n,d) \le \max_{\vec{k}} \left(\sum_{i=1}^{m} c_{\mathbb{Z}}(k_i, d-1) + \sum_{i=1}^{m-1} \min\{k_i, k_{i+1}\} \right)$$

$k_1 + \dots + k_m = n$

$$\| $$

$$\sum_{i=1}^{m-1} \left(\frac{k_i + k_{i+1}}{2} - \frac{|k_i - k_{i+1}|}{2} \right)$$

$$\| $$

$$-\frac{k_1}{2} - \frac{k_m}{2} + \sum_{i=1}^{m} k_i - \sum_{i=1}^{m-1} \frac{|k_i - k_{i+1}|}{2}$$

$$\le \max_{\vec{k}} \left(\sum_{i=1}^{m} c_{\mathbb{Z}}(k_i, d-1) + n - \max k_i \right) \quad \dots$$

Proofs of Thm 1: $c_{\mathbb{Z}}(n,d) \le \lfloor dn - dn^{\frac{d-1}{d}} \rfloor$

II) /elegant/:

assume $\quad \mathrm{vol}_d(\mathrm{P}) = \mathrm{vol}_d(\mathrm{C})$

Minkowski definiton

Brunn–Minkowski inequality

then

$$\mathrm{svol}_{d-1}(\mathrm{P}) = \lim_{\epsilon \to 0+} \frac{\mathrm{vol}_d(\mathrm{P}+\epsilon\mathrm{C}) - \mathrm{vol}_d(\mathrm{P})}{\epsilon}$$

$$\ge \lim_{\epsilon \to 0+} \frac{\left(\mathrm{vol}_d(\mathrm{P})^{\frac{1}{d}} + \mathrm{vol}_d(\epsilon\mathrm{C})^{\frac{1}{d}}\right)^d - \mathrm{vol}_d(\mathrm{P})}{\epsilon} \ge \mathrm{svol}_{d-1}(\mathrm{C})$$

the isoperimetric quotient

$$\frac{\mathrm{svol}_{d-1}(\mathrm{P})^d}{\mathrm{vol}_d(\mathrm{P})^{d-1}} \ge \frac{\mathrm{svol}_{d-1}(\mathrm{C})^d}{\mathrm{vol}_d(\mathrm{C})^{d-1}} \ge (2d)^d$$

and

$$2dn - 2c_{\mathbb{Z}}(n,d) = \mathrm{svol}_{d-1}(\mathrm{P}) \ge 2d\,\mathrm{vol}_d(\mathrm{P})^{\frac{d-1}{d}} = 2dn^{\frac{d-1}{d}}$$

<u>Proof</u> of Thm 2: $c(n,d) \leq \left\lfloor dn - \frac{n^{\frac{d-1}{d}}}{2d^{\frac{d-1}{2}}} \right\rfloor \qquad d \geq 4$

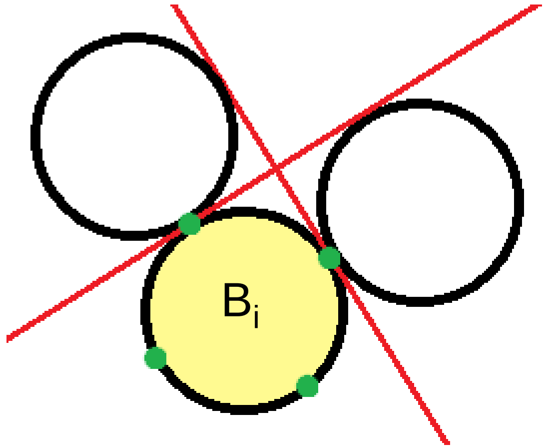
/0/ Notations:

We have the *totally separable* packing **{** $B_i : i \in I$ **}**

where $\mathbf{B_i} = B(c_i, 1) = c_i + B(0,1)$,

further let $r \cdot \mathbf{B_i} := B(c_i, r) = c_i + r \cdot B(0,1)$ for any $r > 0$.

/1/ Recall: *For totally separable packings* maximal number **2*d*** of "kissing points" <=> $\mathbf{t}_1$, ... , $\mathbf{t}_{2d}$ form a regular inscribed cross-polytope**.**



/2/ so $\sqrt{d}\cdot B_i \subseteq \cup_{j\neq i} \sqrt{d}\cdot B_j$

/3/ let m := the number of B_i having $2d$ "kissing points" ($m \le n$)

/4/ so $\mathrm{svol}_{d\text{-}1}\,(\cup_i \sqrt{d}\cdot B_i\,) \le (n\text{-}m)\, d^{(d\text{-}1)/2}\,\mathrm{svol}_{d\text{-}1}(B(0,1))$

/5/ **Osserman**'s (1978) *isoperimetric quotient* **Iq(.)** and -*inequality*

$$\mathrm{Iq}(\mathbf{B}^d) = \frac{\mathrm{svol}_{d-1}\left(\mathrm{bd}(\mathbf{B}^d)\right)^d}{\mathrm{vol}_d(\mathbf{B}^d)^{d-1}} = d^d\mathrm{vol}_d(\mathbf{B}^d) \le$$

$$\le \mathrm{Iq}(\bigcup_i \sqrt{d}\mathbf{B}_i\,) = \frac{\mathrm{svol}_{d-1}\left(\mathrm{bd}(\bigcup \sqrt{d}\mathbf{B}_i\,)\right)^d}{\mathrm{vol}_d(\bigcup \sqrt{d}\mathbf{B}_i\,)^{d-1}}$$

/6/ so

$$n - m \geq \frac{\mathrm{svol}_{d-1}\left(\mathrm{bd}(\bigcup \sqrt{d}\mathbf{B}_i\ )\right)}{d^{\frac{d-1}{2}}\,\mathrm{svol}_{d-1}\left(\mathrm{bd}(\mathbf{B}^d)\right)}$$

$$\cdots$$

$$\geq \quad \frac{n^{\frac{d-1}{d}}}{d^{\frac{d-1}{2}}\,\delta_{\mathbf{sep}}\left(\frac{\sqrt{d}}{2}, d\right)^{\frac{d-1}{d}}}$$

where $\delta_{sep}(R,d)$ **:=** max *density* of R-*separable* ball packings

/7/ finally

$$\mathrm{c}(n,d) \le \frac{1}{2}\left(2dn-(n-m)\right)$$

$$\le dn - \frac{1}{2d^{\frac{d-1}{2}}\,\delta_{\mathrm{sep}}\left(\frac{\sqrt{d}}{2},d\right)^{\frac{d-1}{d}}}\, n^{\frac{d-1}{d}}$$

$$< dn - \frac{1}{2d^{\frac{d-1}{2}}}\, n^{\frac{d-1}{d}} \quad .$$

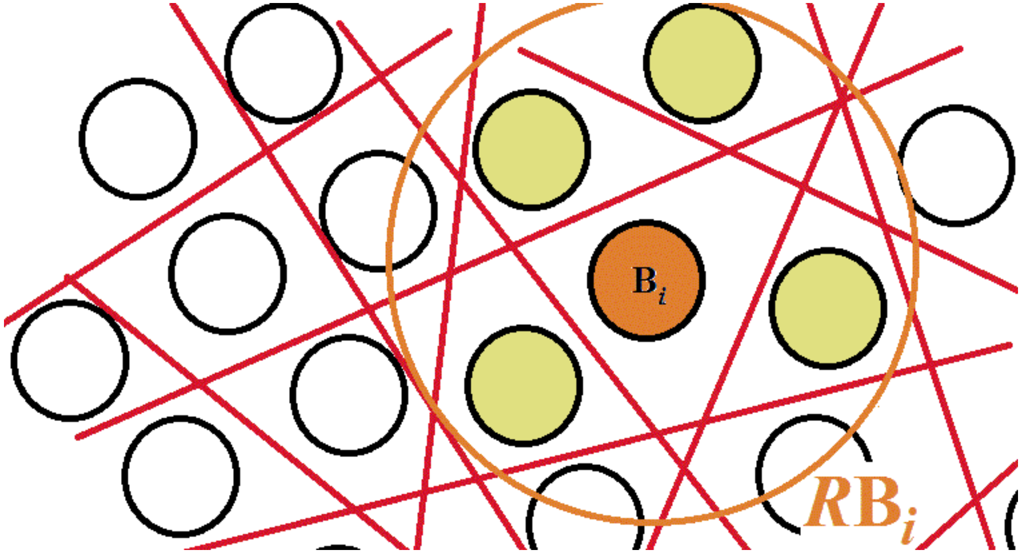
/8/ Def: $\delta_{\text{sep}}(R,d)$ **:=** max *density* of R-*separable* ball packings

$$\delta_{\mathbf{sep}}(R,d)=\sup_{\mathcal{P}_{\mathbf{sep}}}\left(\limsup_{\lambda\to+\infty}\frac{\sum_{\mathbf{B}_i\subset\mathbf{Q}_\lambda}\mathrm{vol}_d(\mathbf{B}_i)}{\mathrm{vol}_d(\mathbf{Q}_\lambda)}\right)$$

and a (finite or infinite) packing of balls $P=\{B_i : i \in I\}$ is called

(locally) ***R-separable*** iff for each $i \in I$ the finite packing

$\{B_j : B_j \subseteq RB_i\}$ is totally separable (in RB_i) :



/9/ Lemma: If $\{B_i : i \le n\}$ is an ***R-separable*** ball packing, then

$$\delta_{\mathbf{sep}}(R,d) \geq \frac{n\,\mathrm{vol}_d(\mathbf{B}^d)}{\mathrm{vol}_d\left(\bigcup_{i=1}^{n} 2R\mathbf{B}_i\right)}$$

<u>Proof</u> of Thm 3: $c(n,3) < \lfloor 3n - 1.346\, n^{\frac{2}{3}} \rfloor$

... similar but longer argument ... .

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Thanks for your attention !