Approximate Steiner Trees

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Geometry and Symmetry, Veszprém
3 July 2015

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Happy Birthday, Karoly and Egon!

K. Bezdek's contribution to Steiner minimal trees

Periodica Mathematica Hungarica Vol. 24 (2), (1992), pp. 119–122

PROBLEM. Find the smallest positive real number $f(d)$ (if it does not exist, then let $f(d) = +\infty$) such that for every centrally symmetric convex body $\mathbf{B}$ of $\mathbf{E}^d$, $d \geq 3$ with center O and distance function $d_{\mathbf{B}}$ and for every $l > f(d)$ there exist light-sources $L_1, L_2, \ldots, L_n$ of $\mathbf{E}^d \backslash \mathbf{B}$ which illuminate $\mathbf{B}$ moreover, $\sum_{i=1}^{n} d_{\mathbf{B}}(O, L_i) = l$. Is $f(d) < +\infty$?

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Problem. **Find the smallest positive real number $f(d)$ (if it does not exist, then let $f(d) = +\infty$) such that for every centrally symmetric convex body $\mathbf{B}$ of $\mathbf{E}^d$, $d \geq 3$ with center O and distance function $d_\mathbf{B}$ and for every $l > f(d)$ there exist light-sources $L_1, L_2, \ldots, L_n$ of $\mathbf{E}^d \setminus \mathbf{B}$ which illuminate $\mathbf{B}$ moreover, $\sum_{i=1}^{n} d_\mathbf{B}(O, L_i) = l$. Is $f(d) < +\infty$?**

It turns out that Bezdek's $f(d)$ is an upper bound for the maximum degree of a Steiner minimal tree in a d-dimensional normed space!

For more, see:

S, Quantitative illumination of convex bodies and vertex degrees of geometric Steiner minimal trees, Mathematatika, ***52*** *(2005), 47–52.*

Overview

The Euclidean Steiner problem

Fermat and Torricelli

Melzak algorithm

The Euclidean Steiner problem is difficult

Approximate Steiner trees
- Previous results
- New results

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The Euclidean Steiner problem

Given a finite set N of points (terminals) in Euclidean d-space, find a shortest connected set containing them.

A solution is necessarily a tree $T = (V, E)$ with $N \subseteq V$ such that

1. the angle between two edges with the same endpoint is $\geq 120^\circ$
2. all degrees are at most 3
3. the degree of each vertex in $V \setminus N$ (Steiner points) equals 3
4. if a vertex has degree 3, all angles are 120° and the incidenct edges are coplanar

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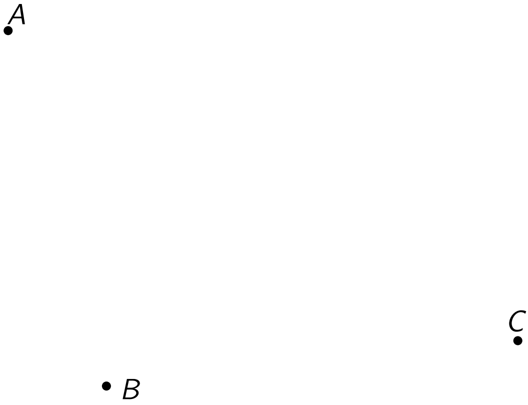
Any tree $T = (V, E)$ with $N \subseteq V$ that satisfies 1. to 4. above is called a Steiner tree of N.

A Steiner tree is full if each terminal has degree 1.

Usually we can reduce problems to the case of full Steiner trees.

Overview

3 terminals



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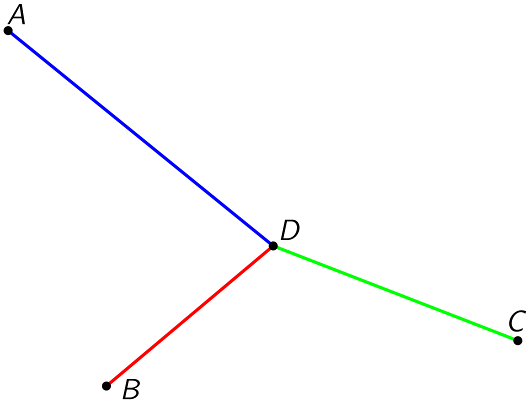

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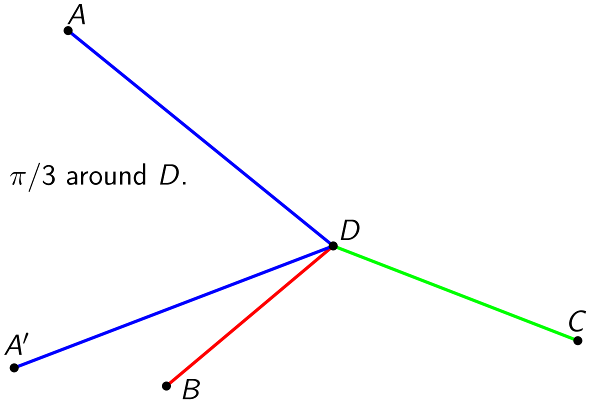

Rotate AD by $\pi/3$ around D.

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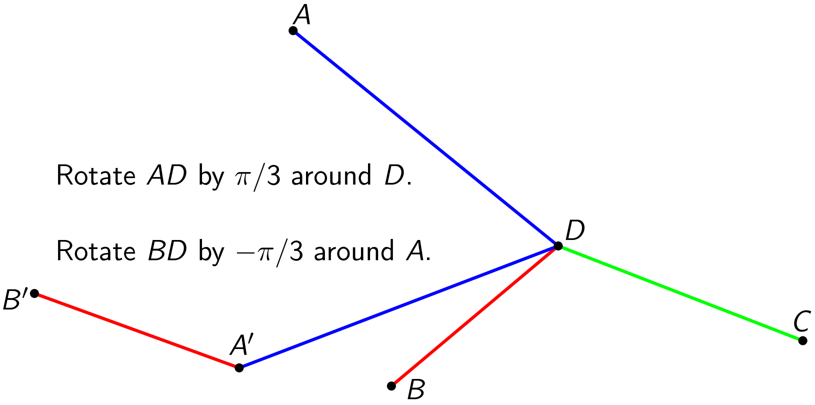

Rotate AD by $\pi/3$ around D.

Rotate BD by $-\pi/3$ around A.

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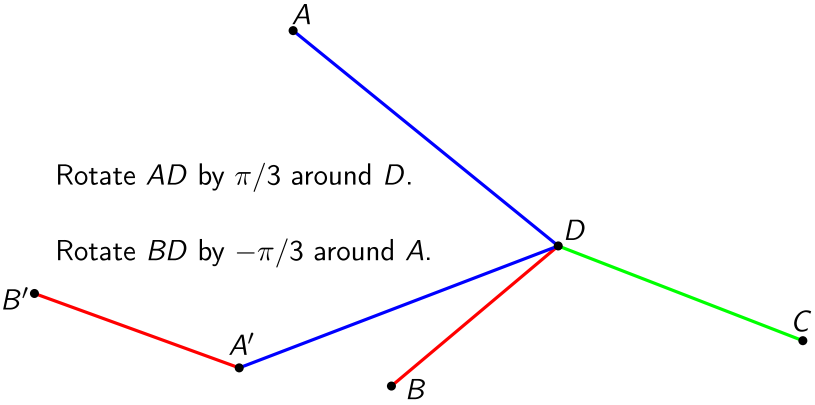

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Any candidate tree can be unfolded to a broken line.

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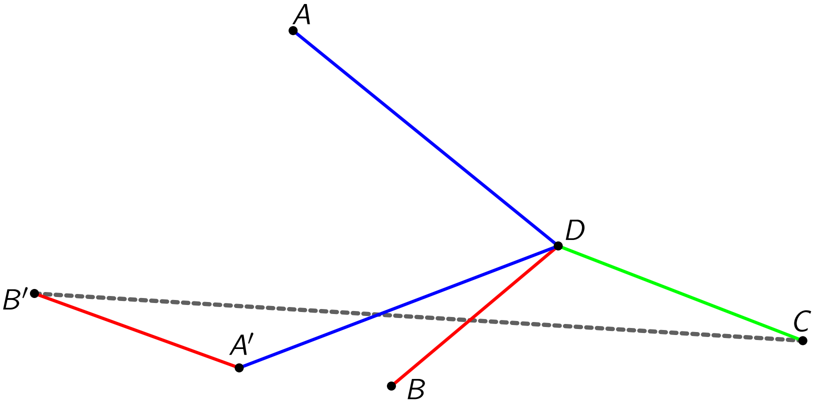

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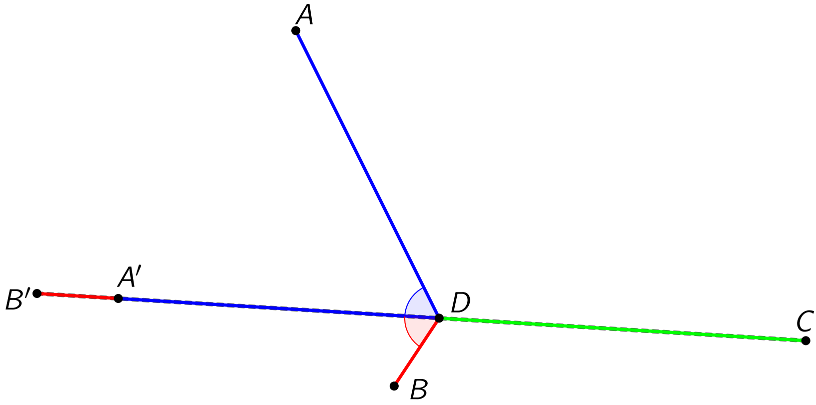

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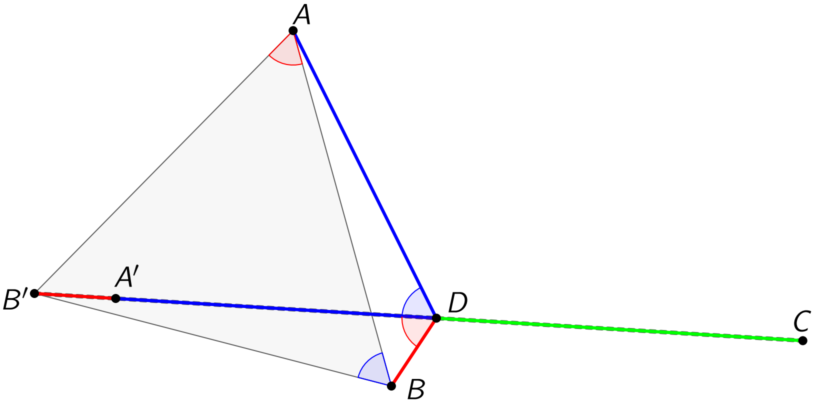

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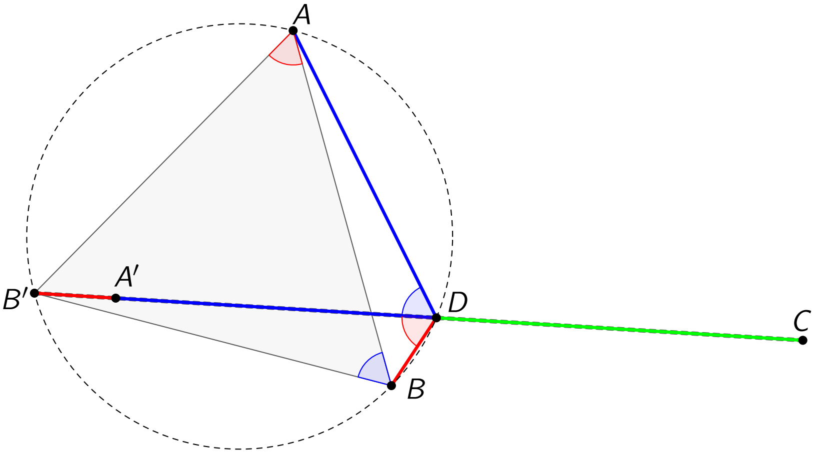

Any candidate tree can be unfolded to a broken line.

The Steiner point D is the intersection of line $B'C'$ and the circumcircle of the equilateral triangle.

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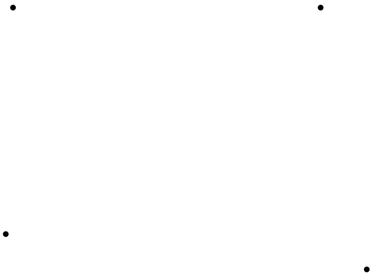

Given a graph structure $\equiv$ Steiner topology,

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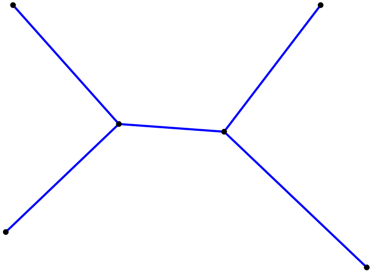

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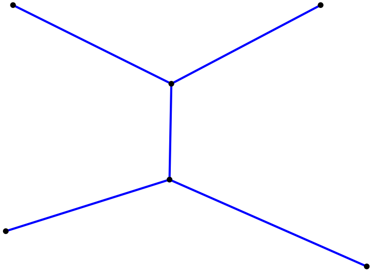

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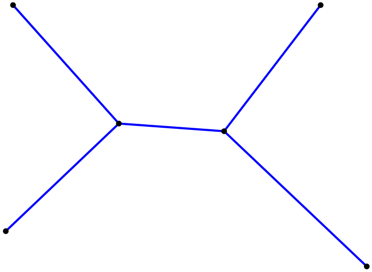

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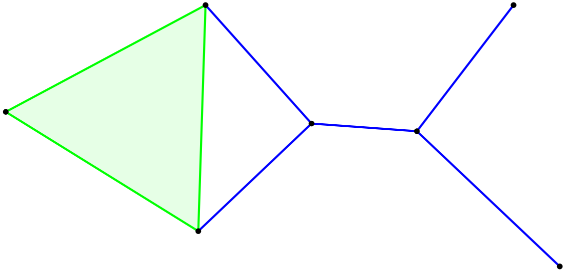

Given a graph structure $\equiv$ Steiner topology, the Melzak algorithm (1965) finds a shortest Steiner tree of a given topology if non-degenerate

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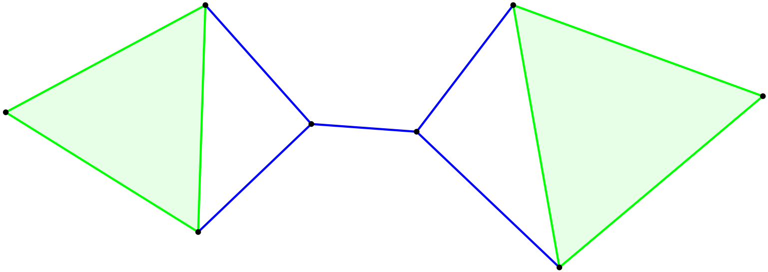

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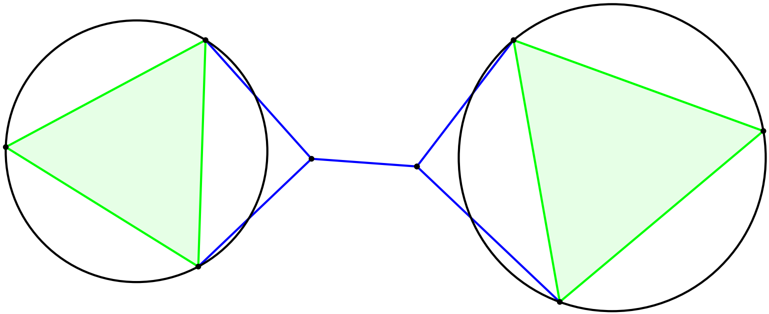

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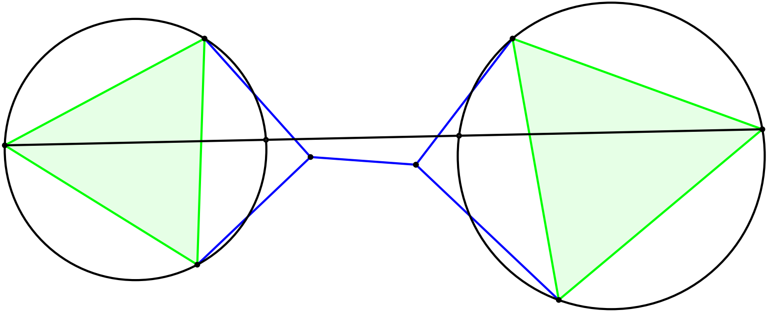

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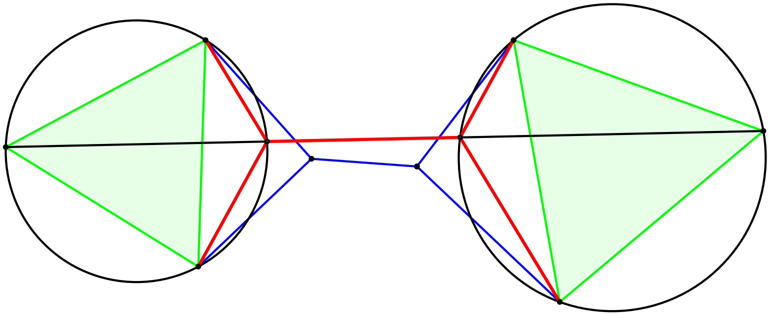

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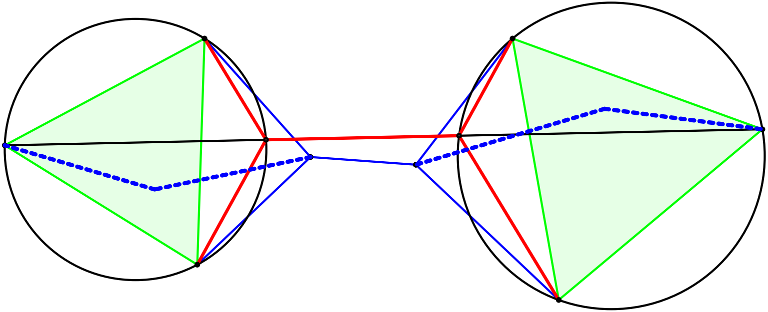

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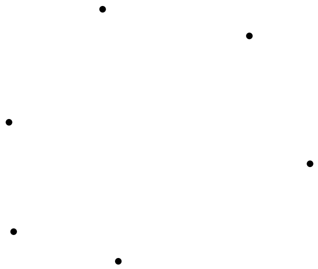
and again note the **unfolding** of the original candidate tree

The Melzak algorithm



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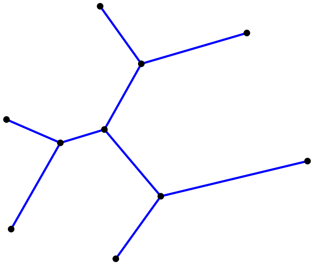

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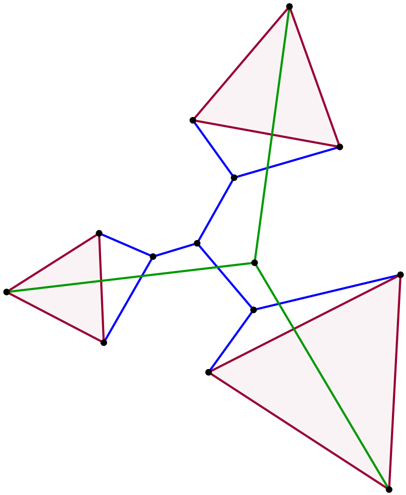

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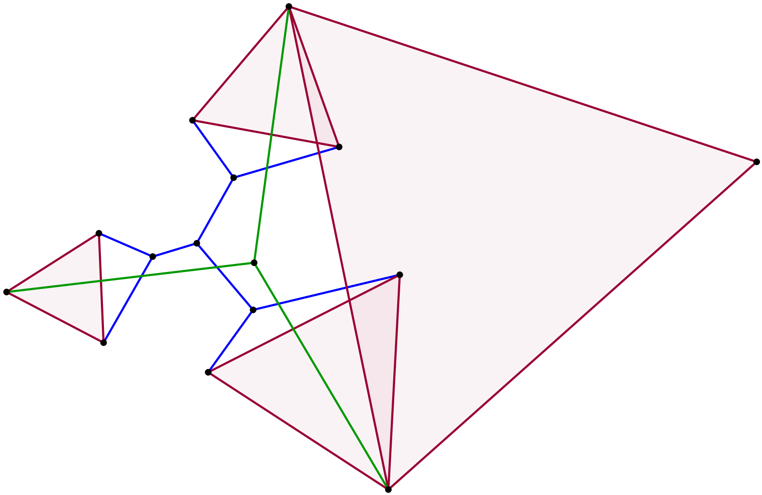

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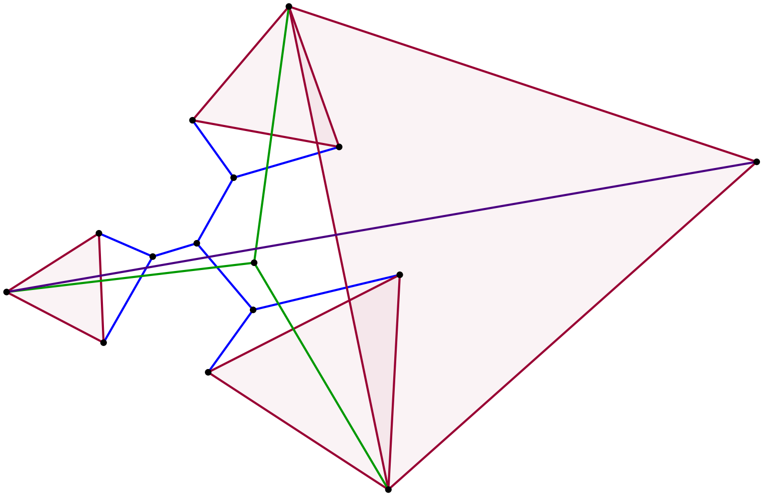

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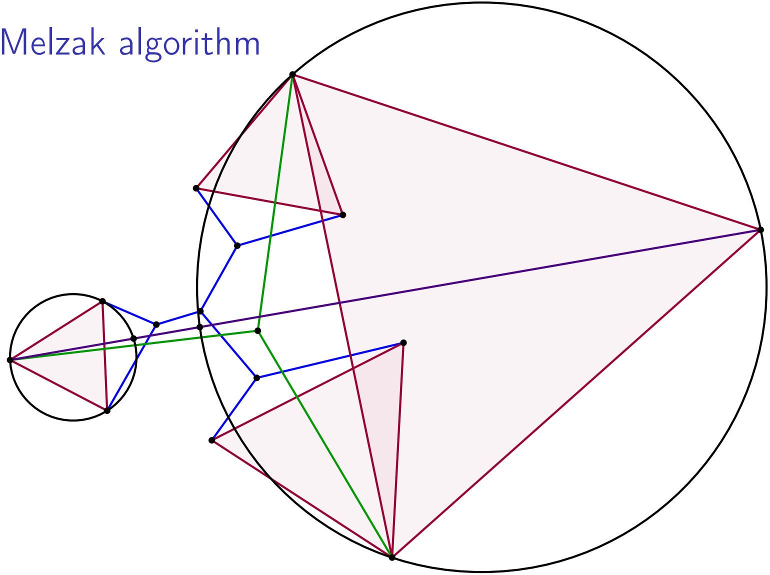

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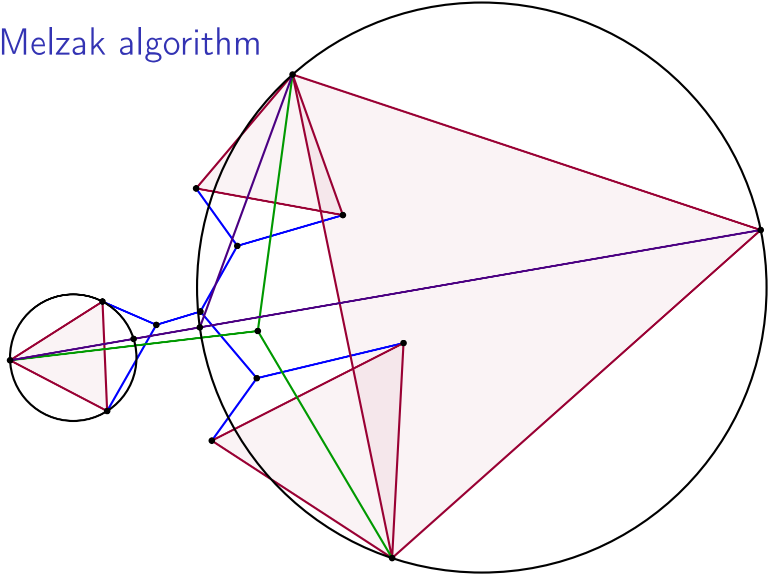

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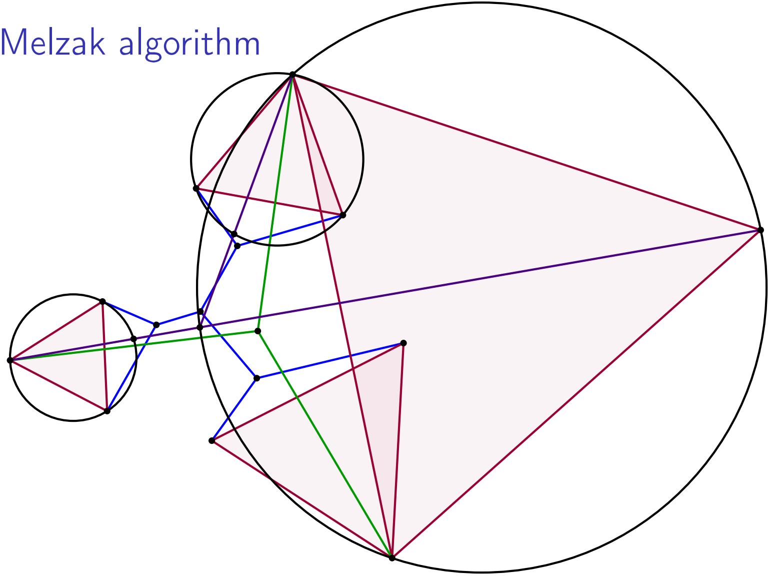

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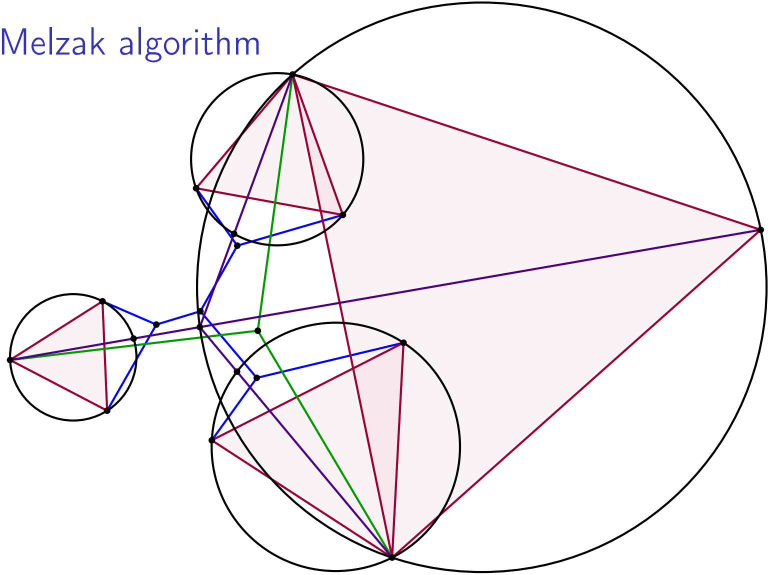

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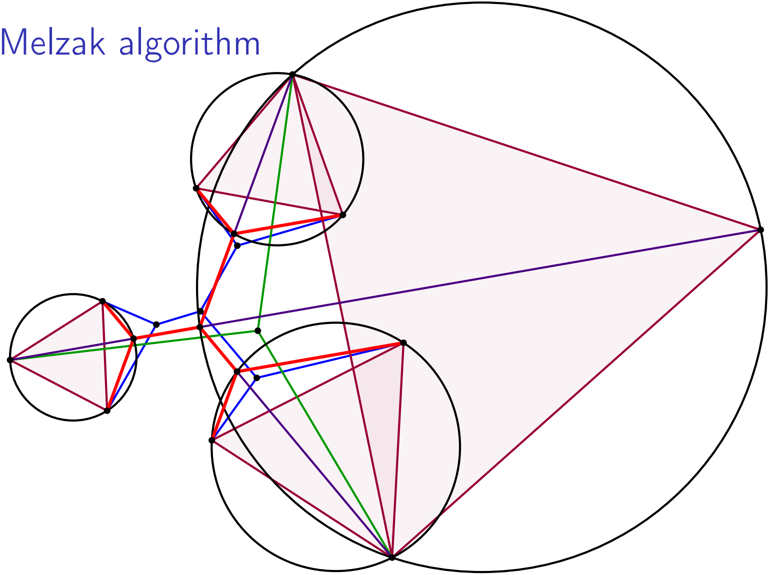

As modified by Hwang (1976), the Melzak algorithm has linear time complexity, assuming constant time real arithmetic operations $(+, -, \times, \div, \surd)$. **There is again an unfolding.**

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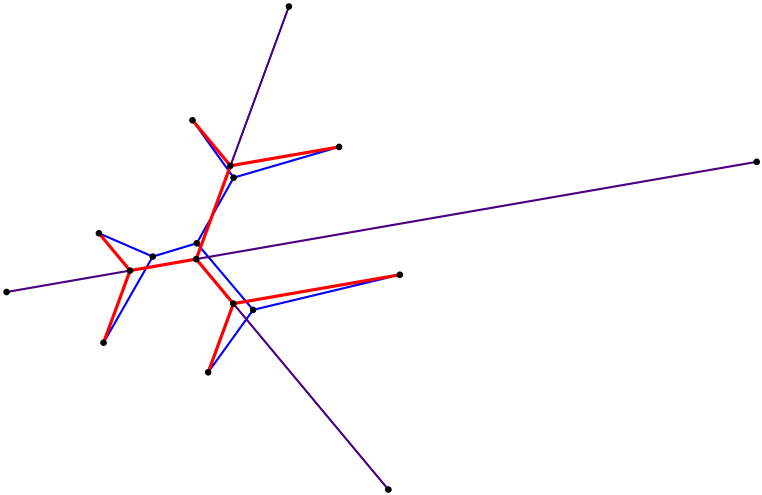

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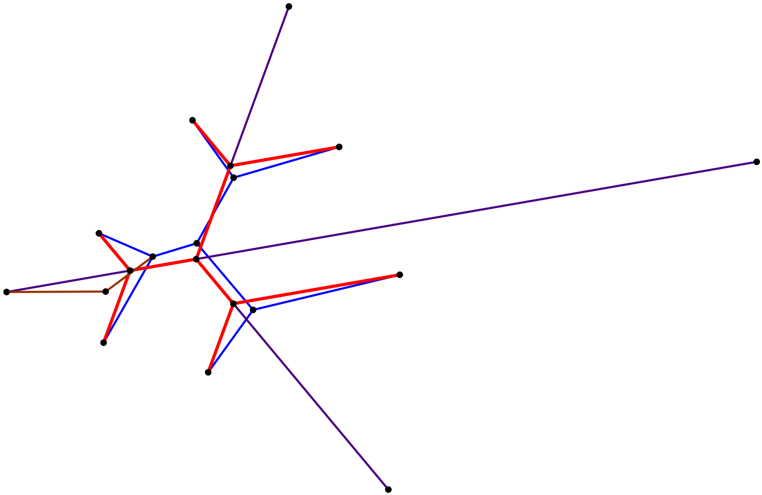

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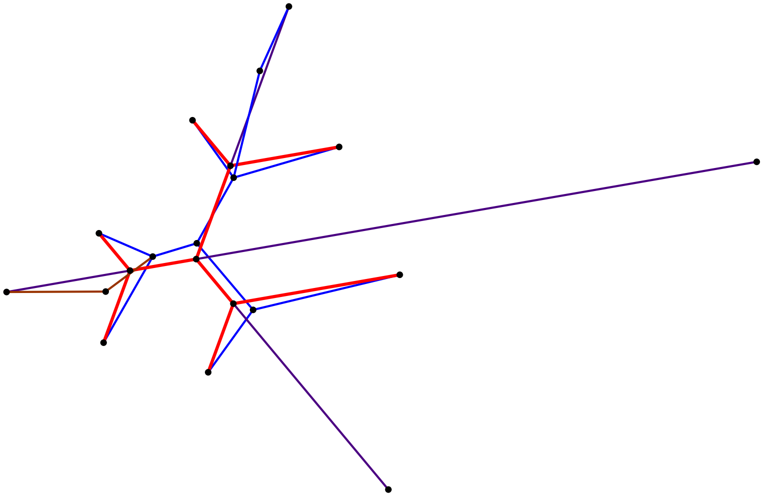

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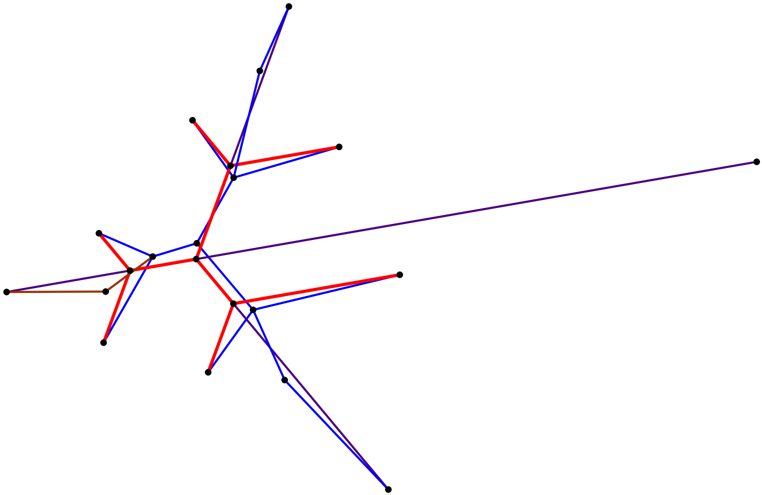

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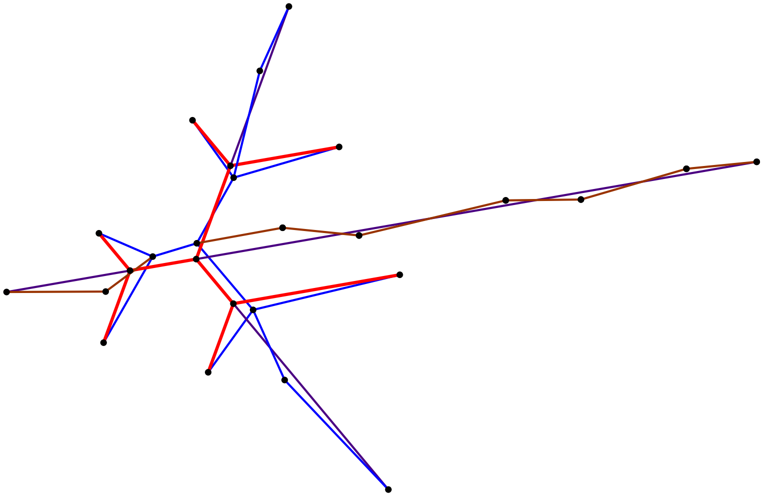

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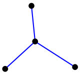
In the plane

- there is a compass-and-ruler construction (Melzak algorithm), but there are superexponentially many topologies



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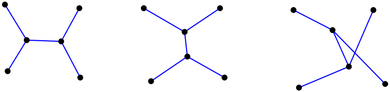

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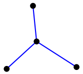

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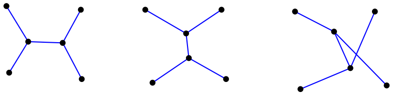

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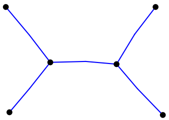

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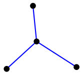

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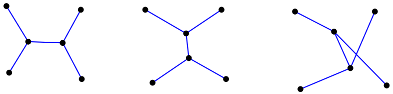

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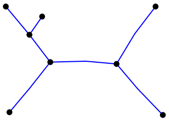

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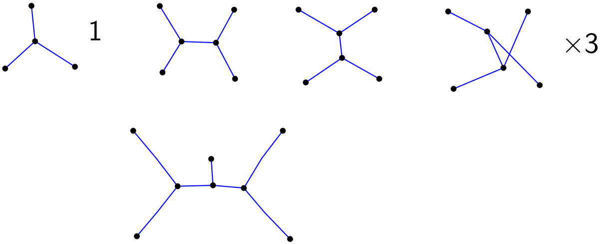

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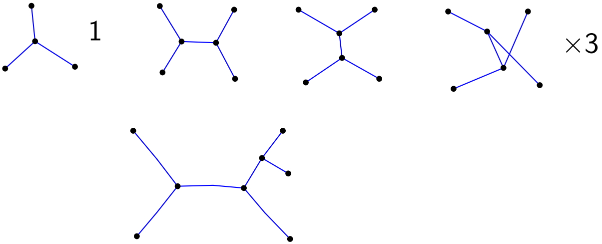

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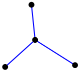

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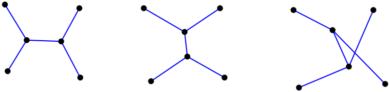

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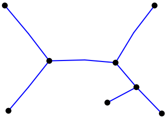

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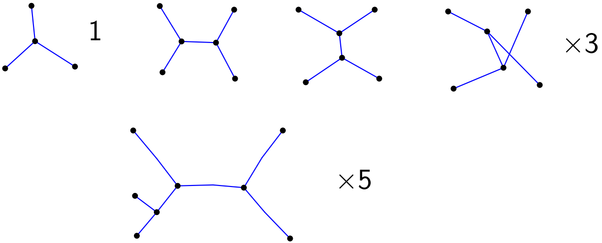

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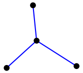

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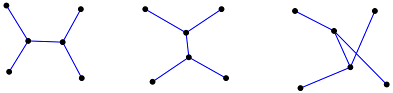

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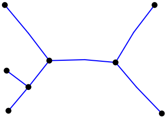

$\times 5$

6 terminals: $1 \times 3 \times 5 \times 7$ (full) topologies, etc.

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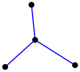

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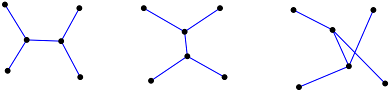

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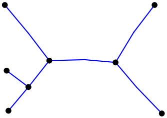

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- NP-hard (Garey–Graham–Johnson 1977)

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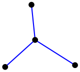
- there is a compass-and-ruler construction (Melzak algorithm), but there are superexponentially many topologies



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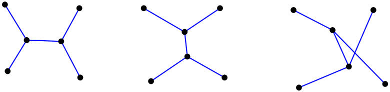

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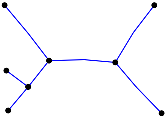

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6 terminals: $1 \times 3 \times 5 \times 7$ (full) topologies, etc.

- NP-hard (Garey–Graham–Johnson 1977)
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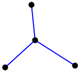

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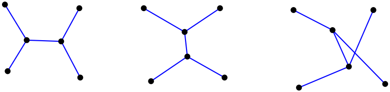

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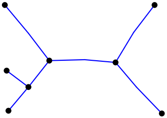

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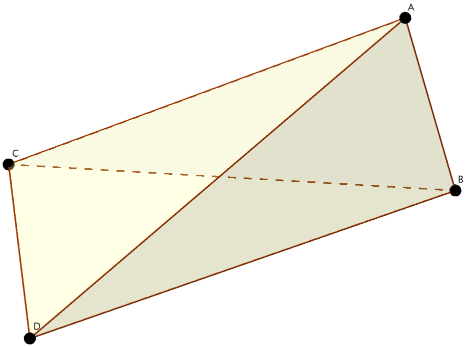
In 3-space the problem becomes non-constructable, even unsolvable by radicals (Smith 1992, Mehlhos 2000, Rubinstein et al. 2004)



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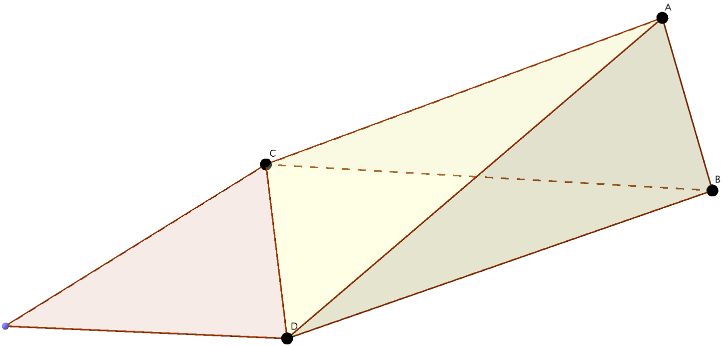

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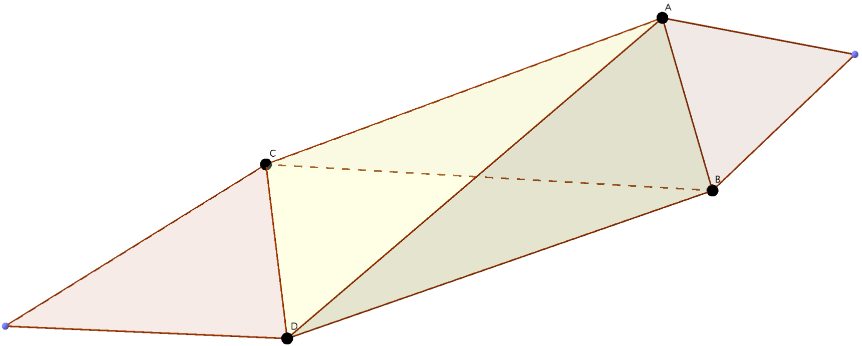

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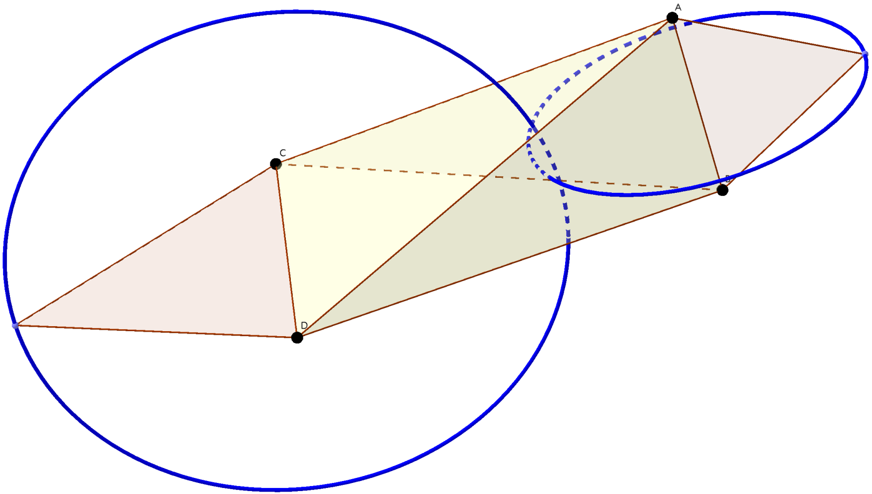

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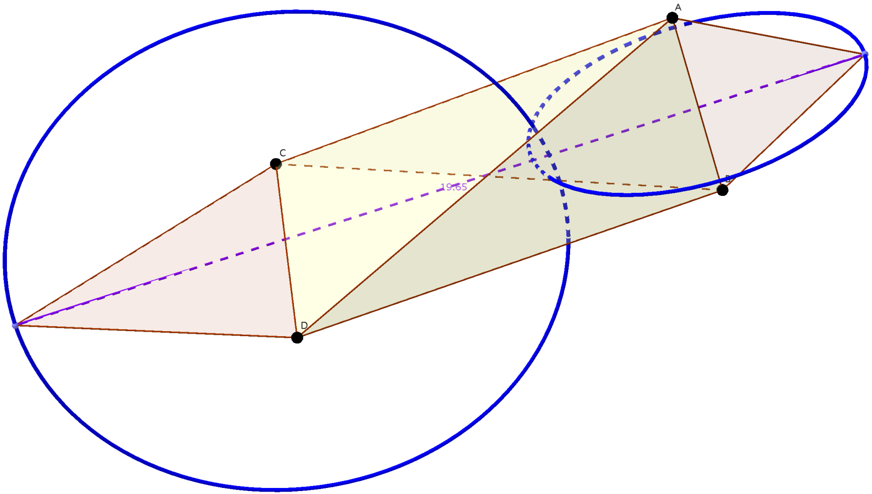

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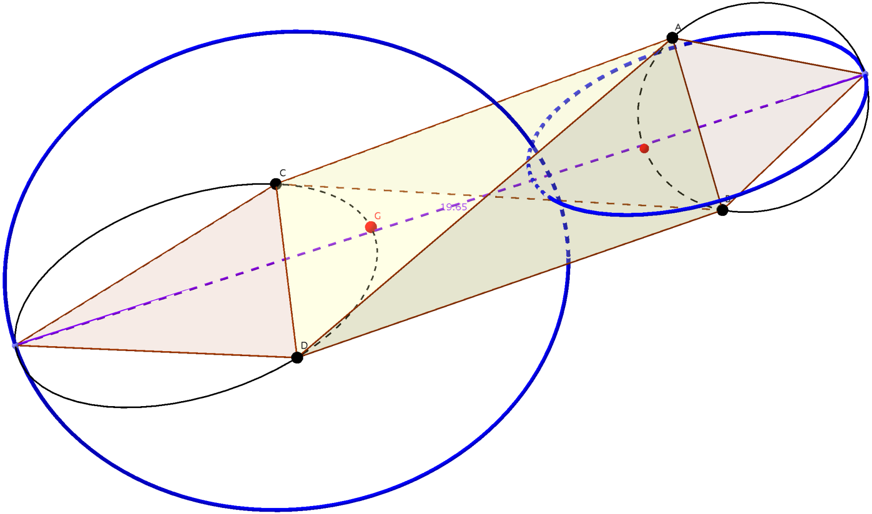

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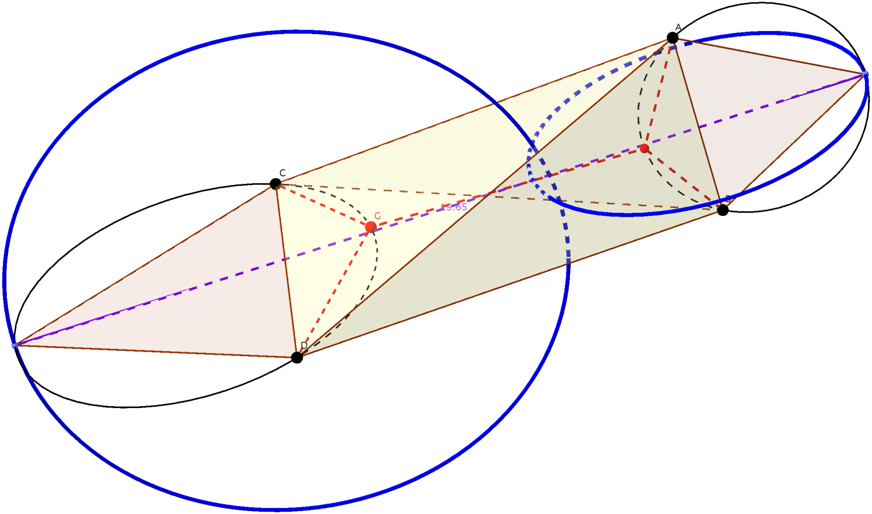

Overview

The Euclidean Steiner problem

Fermat and Torricelli

Melzak algorithm

The Euclidean Steiner problem is difficult

Approximate Steiner trees

- Previous results
- New results

Approximate Steiner trees

Thus, numerical methods are used, giving an approximate solution.

How far is its length from optimal?

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Definition (Rubinstein–Weng–Wormald 2006)

Given a set N of points (terminals) in $\mathbb{R}^d$, an approximate Steiner tree on N is a tree $T = (V, E)$ such that

- $N \subseteq V \subset \mathbb{R}^d$,
- the degree of each terminal is at most 3, and
- the degree of each vertex in $V \setminus N$ (pseudo-Steiner points) has degree 3.

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Given $\varepsilon \geq 0$, an approximate Steiner tree on N is called ε-approximate if at each pseudo-Steiner point, the three angles are in the range $[2\pi/3 - \varepsilon, 2\pi/3 + \varepsilon]$, and at each terminal, the angles are at least $2\pi/3 - \varepsilon$.

(A 0-approximate Steiner tree is just a Steiner tree.)

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Problem

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, find an upper bound for the relative error in the length of an ε-approximate Steiner tree, compared to a shortest Steiner tree of the same topology*.*

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Define $\quad F_d(\varepsilon, n) = \sup \left\{ \dfrac{L(T) - L(S(T))}{(L(S(T))} \; : \; T \in \mathcal{A}_\varepsilon^d(n) \right\}.$

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They only considered dimensions 3 and higher:

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What about the plane?

All upper bounds for $d = 3$ still hold.

$F_2(\varepsilon, n) > c\varepsilon^2$ is trivial.

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Joint work with Charl Ras and Doreen Thomas (Melbourne)

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Thus, we cannot expect any stronger conjecture for the plane.

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Nevertheless, the conjecture is still open for "large" ε, even for $\varepsilon = 1^\circ$, say. The best upper bound we have in the plane is:

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If $\varepsilon \leq \pi/6$ and $n \geq 2$ then $F_2(\varepsilon, n) \leq 2n - 4$.

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Note the following cautionary lower bound:

Proposition (RST, based on Rubinstein et al. 2006)

For all $d \geq 2$, $F_d(\pi/3, n) = \Omega(\log n)$.

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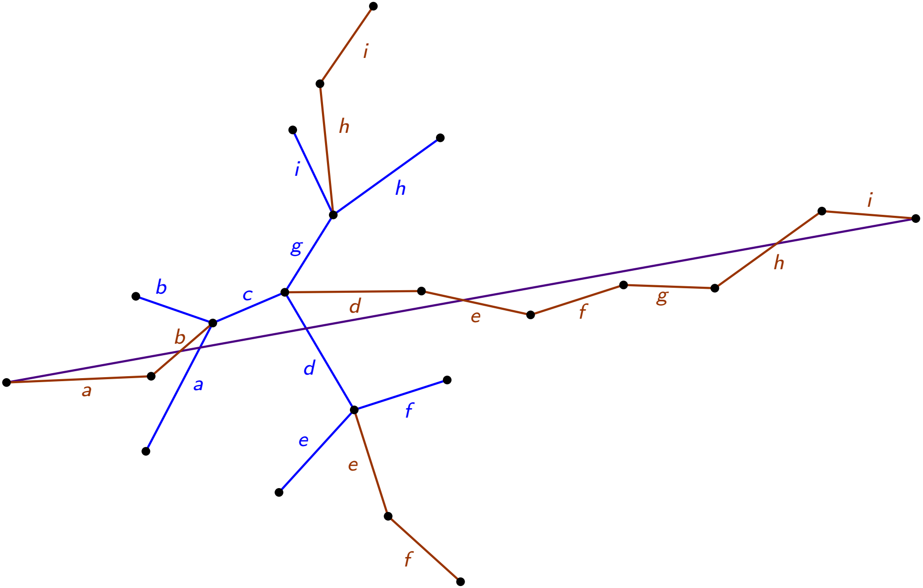
Proof.

Consider an ε-approximate tree T in $\mathbb{R}^2$.

1. Unfold T into a path P using the Melzak algorithm.
2. Estimate the turns at each vertex of P in terms of the deviations from $2\pi/3$ of the angles at pseudo-Steiner points.
3. Estimate the length of P in terms of the turns, using an old result of Erhard Schmidt, related to the Cauchy Arm Lemma.

We already did 1. ∎

2. Estimate the turns in the path



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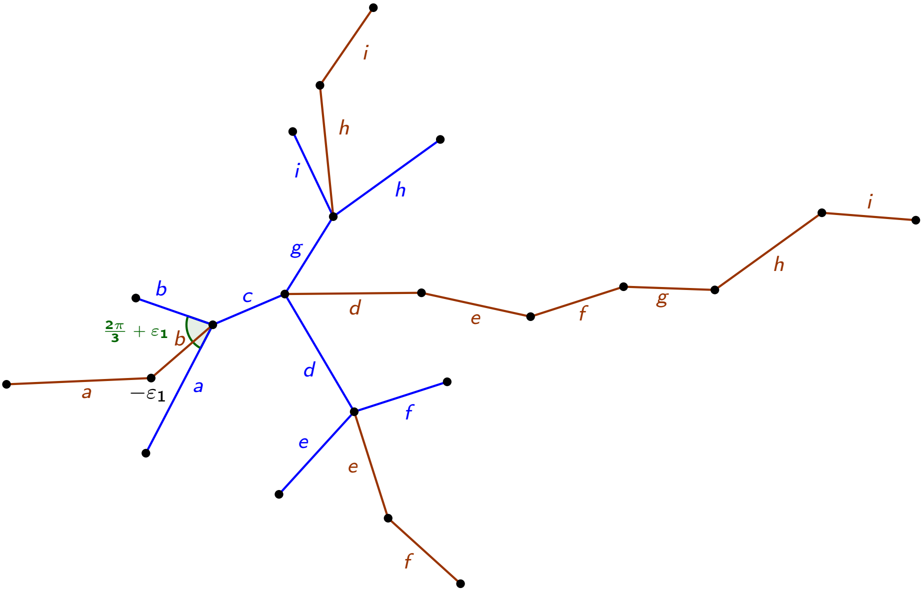

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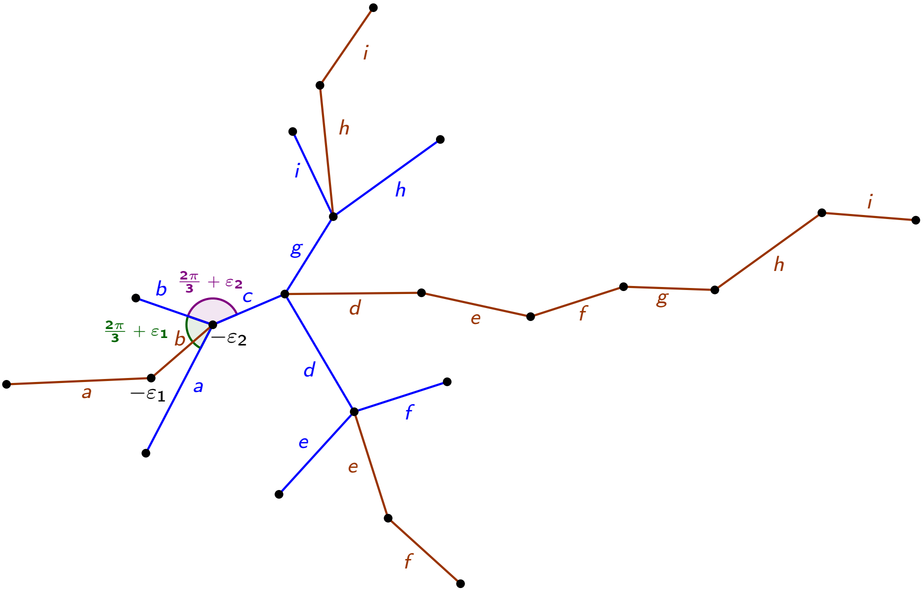

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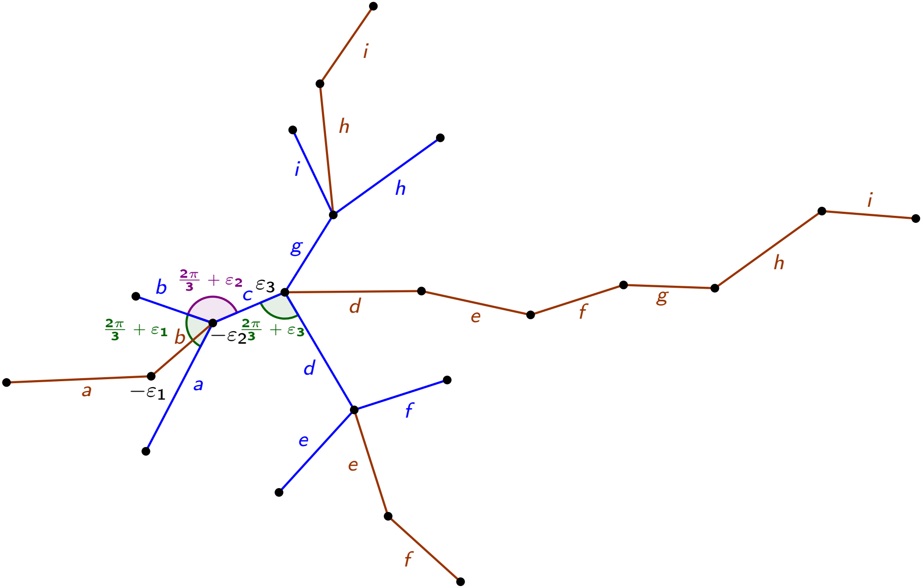

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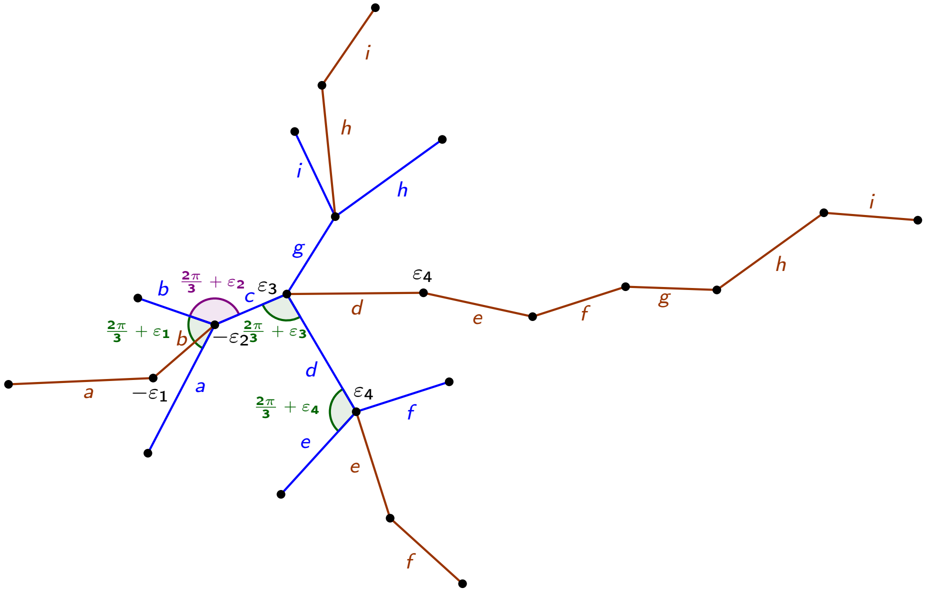

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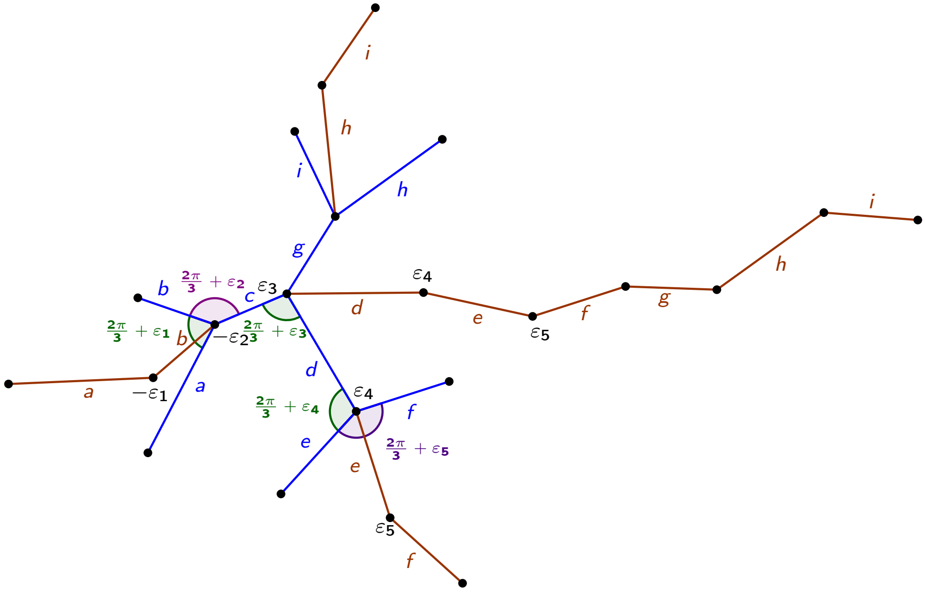

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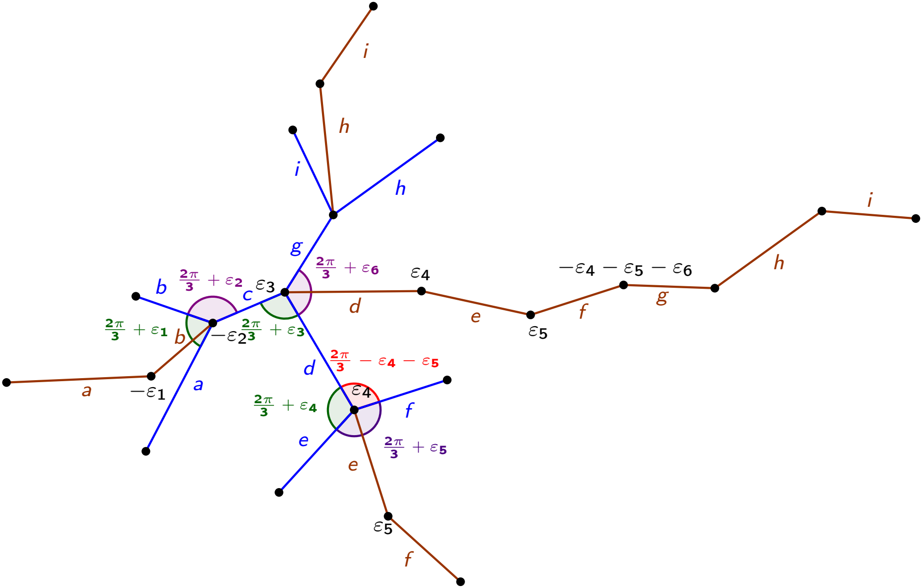

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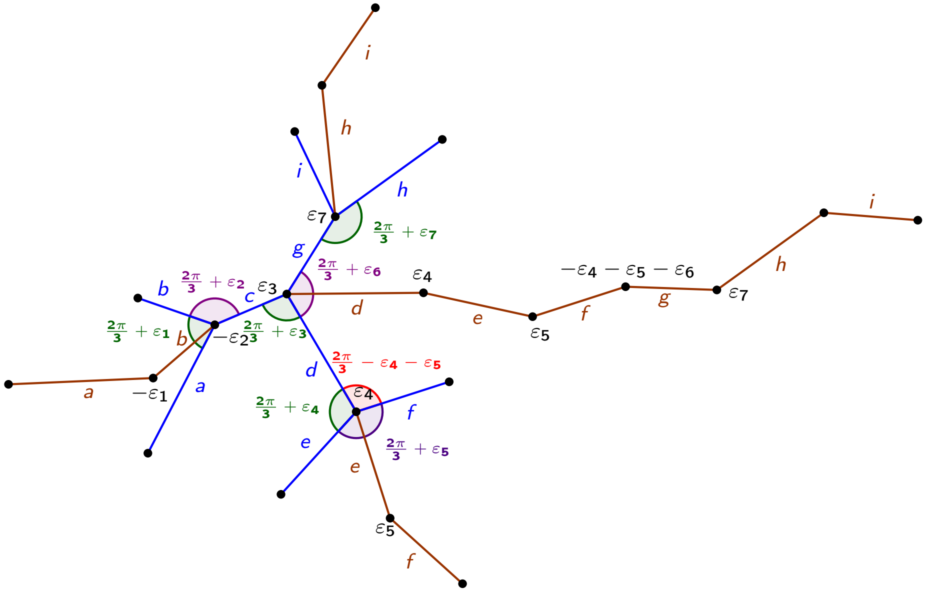

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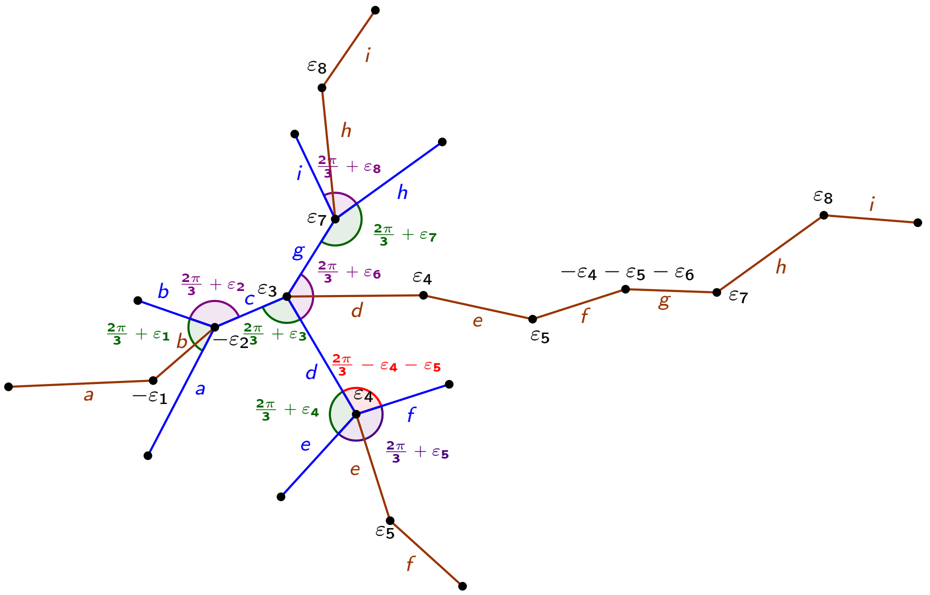

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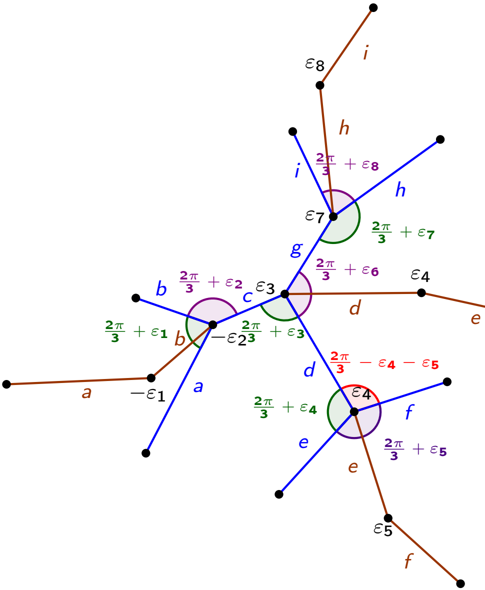
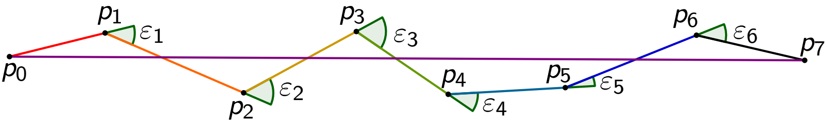


$|\varepsilon_i| \leq \varepsilon$

Note that the sum of turns in any subpath is at most

$2\varepsilon \times$ number of Steiner points.

3. Bounding the length of a path



Lemma (E. Schmidt 1925)

Consider a planar polygonal line $p_0 p_1 \dots p_n$ with turn ε_i at p_i $(i = 1, \dots, n-1)$. Let

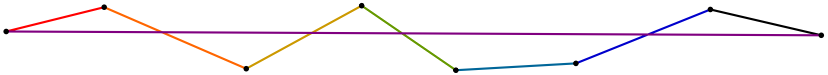
$$\kappa = \max_{1 \le i < j \le n-1} \left| \sum_{t=i}^{j} \varepsilon_t \right| .$$

If $\kappa < \pi$, then

$$\frac{\sum_{i=1}^{n-1} |p_i p_{i+1}|}{|p_0 p_n|} \le \frac{1}{\cos \kappa/2}.$$

3. Bounding the length of a path

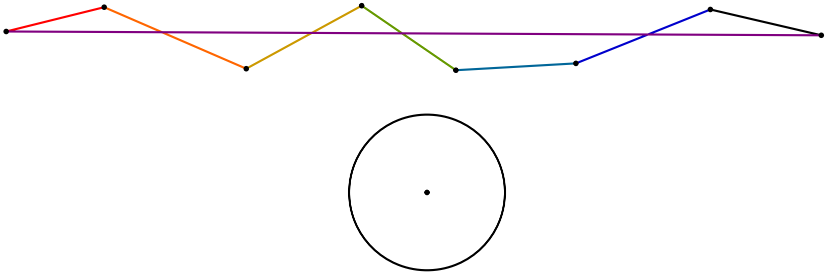
Proof.



3. Bounding the length of a path

Proof.

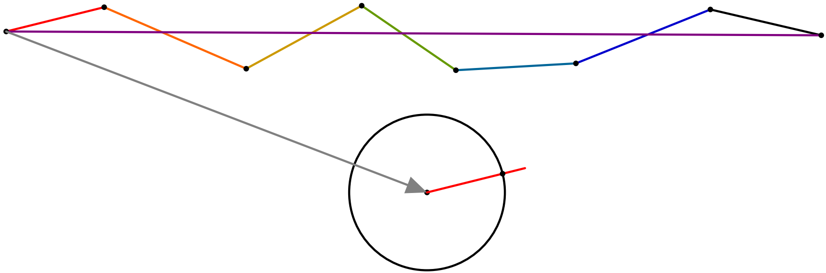
Translate all edges to the origin.



3. Bounding the length of a path

Proof.

Translate all edges to the origin.



3. Bounding the length of a path

Proof.

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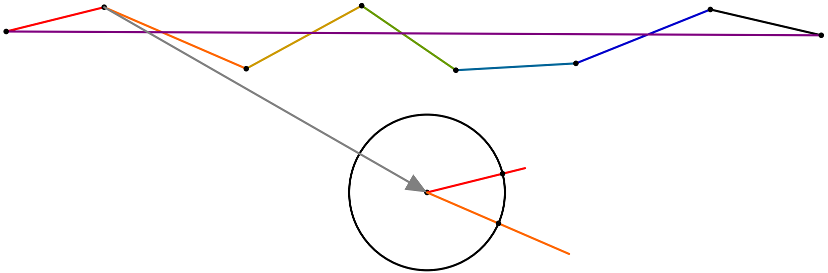
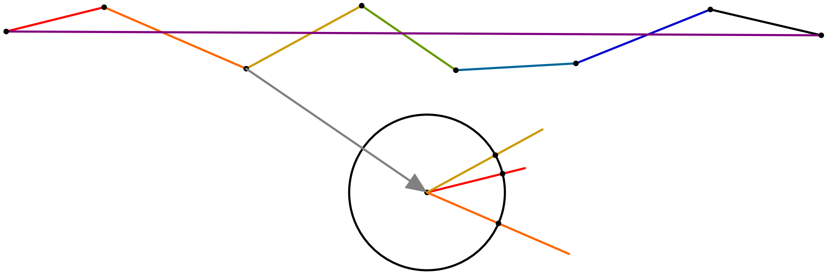


3. Bounding the length of a path

Proof.

Translate all edges to the origin.



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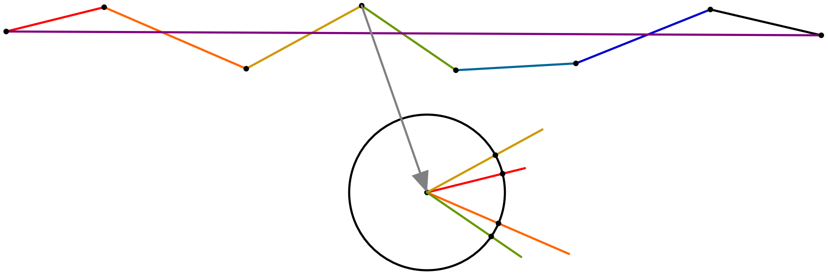
Proof.

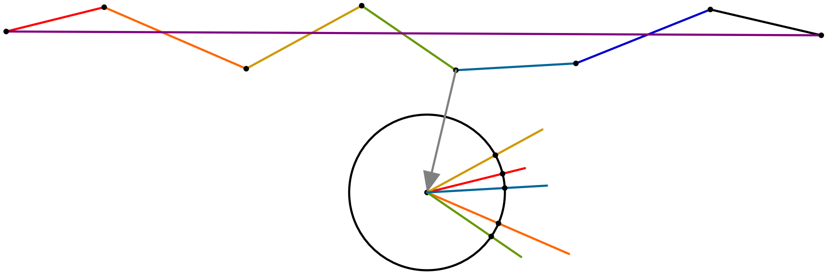
Translate all edges to the origin.



3. Bounding the length of a path

Proof.

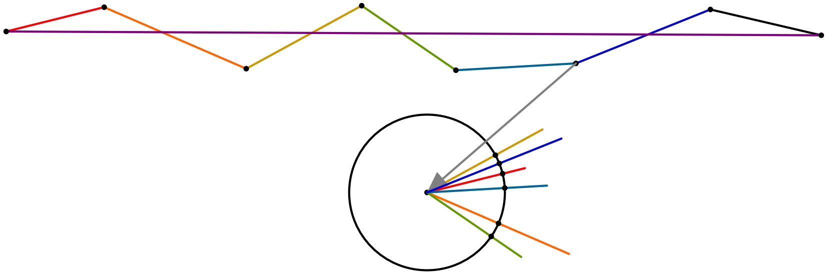
Translate all edges to the origin.



3. Bounding the length of a path

Proof.

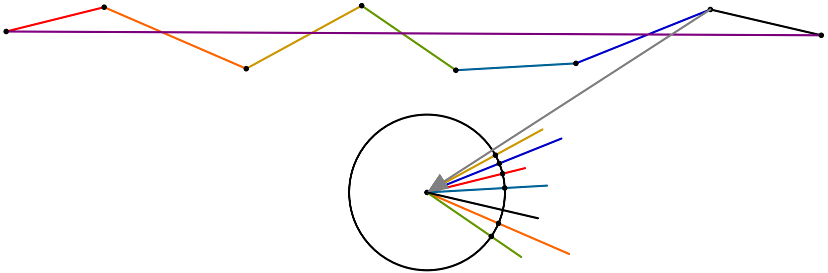
Translate all edges to the origin.



3. Bounding the length of a path

Proof.

Translate all edges to the origin.



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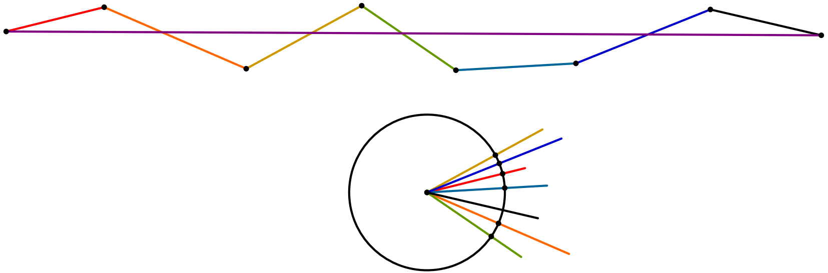
Proof.

Translate all edges to the origin.

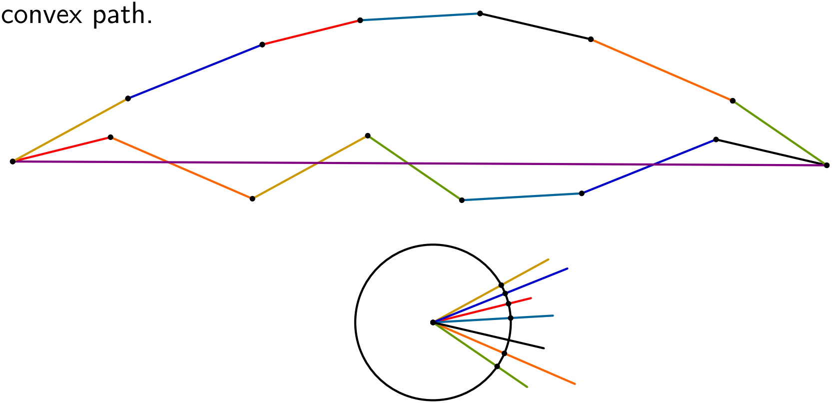


3. Bounding the length of a path

Proof.

Translate all edges to the origin.

Rearrange to form a convex path.



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Proof.

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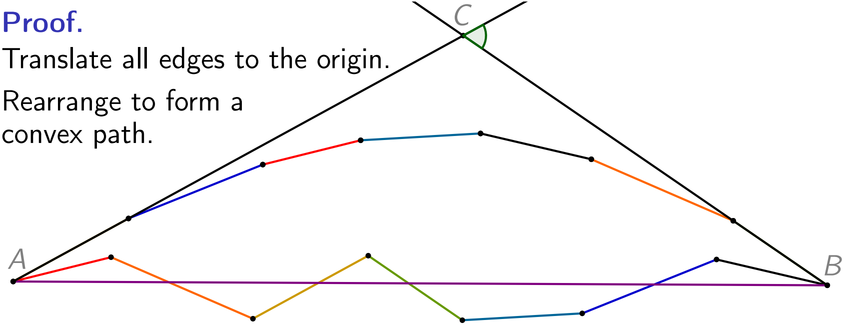
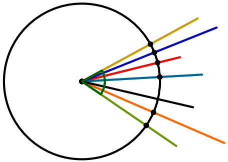
Rearrange to form a convex path.



$$\frac{\text{Length of path}}{|AB|} \le \frac{|AC| + |CB|}{|AB|}$$



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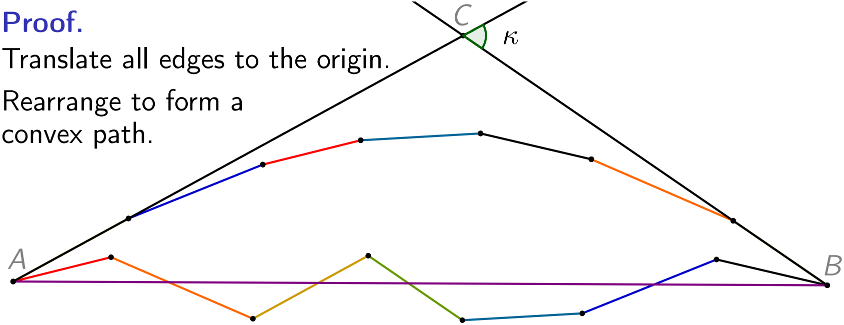
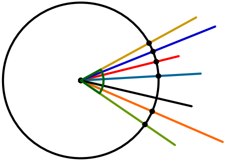


$$\frac{\text{Length of path}}{|AB|}$$

$$\leq \quad \frac{|AC| + |CB|}{|AB|}$$



$$\kappa = \max_{1 \leq i < j \leq n-1} \left| \sum_{t=i}^{j} \varepsilon_t \right|$$

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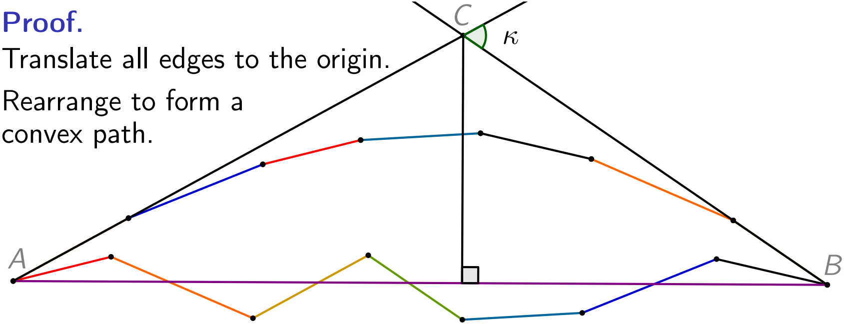
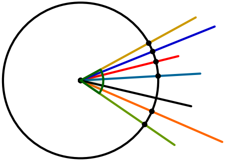
Proof.

Translate all edges to the origin.

Rearrange to form a convex path.



$$\frac{\text{Length of path}}{|AB|} \leq \frac{|AC| + |CB|}{|AB|} \leq \frac{1}{\cos \kappa/2}. \qquad \blacksquare$$



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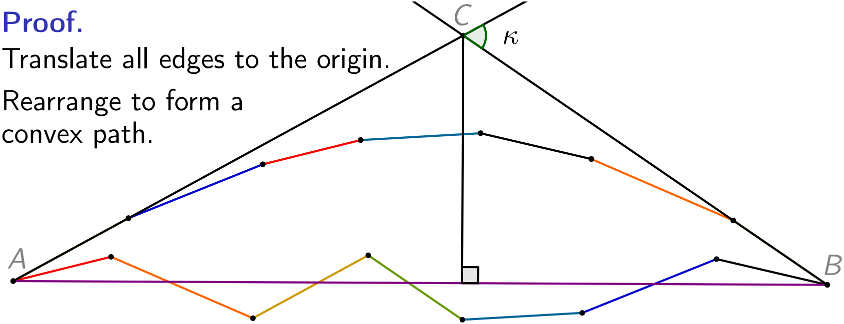
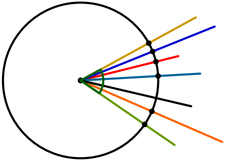


$$\frac{\text{Length of path}}{|AB|} \le \frac{|AC| + |CB|}{|AB|} \le \frac{1}{\cos \kappa/2}. \qquad \blacksquare$$



$$\kappa = \max_{1 \le i < j \le n-1} \left| \sum_{t=i}^{j} \varepsilon_t \right|$$

Thank you for your attention.