The degree-diameter problem for vertex-transitive graphs and finite geometries

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- Is it true that $n(d,k) > (1-\varepsilon)M(d,k)$ for all $d > d_\varepsilon$, $k > k_\varepsilon$?

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Consequence: Bad news for attempts to construct Γ by coverings!

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So, the above Cayley graphs arise from generalised triangles with polarity!

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$$\Omega = \{[0,1,0,0]\} \cup \{[1, f(x,y), x, y];\ x, y \in F\}$$

invariant is the Suzuki group $Sz(q) = {}^2B_2(q)$, and G is the subgroup of $Sz(q)$ that stabilises the point $[0,1,0,0]$.

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If a graph Γ *is isomorphic to a Cayley graph* $C(G,X)$*, then, for any prime* p*, the number of oriented closed walks of length* p *in* Γ*, based at a fixed vertex, is congruent mod* p *to the number of generators of order* p *in* X*.*

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Digraphs $\Gamma_{\delta,k}$: vertices are k-strings of distinct symbols from a set L, $|L| = \delta + 1$; $3 \leq k \leq \delta$. Any vertex $v = x_1 x_2 \ldots x_k$ sends a dart into each $v_y = x_2 \ldots x_k y$ where $y \in L \setminus \{x_1, \ldots, x_k\}$ and also, for $1 \leq i \leq k-1$, into each v_i obtained from v by moving x_i to the right end of the string.

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F-M-Ch are Cayley graphs only in rare cases. [Staneková-Ždímalová '10]

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Symmetrically yours, JŠ