On homothetic copies of a convex body

joint works of subsets of:
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Bezdek–Pach Conjecture

K — a convex body in $\mathbb{R}^d$.
Translate: $t + K$, where $t \in \mathbb{R}^d$.
Homothet: $t + \lambda K$, where $t \in \mathbb{R}^d$ and $\lambda > 0$.

Klee's question (going back to Erdős), '60:
Maximum number of pairwise touching translates of K?
Danzer and Grünbaum, '62:
2^d, and equality exactly for parallelotopes.

Bezdek–Pach Conjecture, '88:
Maximum number of pairwise touching homothets of K is also $\leq 2^d$.
N, '06:
Maximum number of pairwise touching homothets of K is $< 2^{d+1}$.
Zs. Lángi, N, '09: If $K = -K$ then the
maximum number of pairwise touching homothets of K is $< \frac{3}{2} 2^d$.

N, '06:

Maximum number of pairwise touching homothets of K is $< 2^{d+1}$.

Idea of the proof:

1. Veronese–like mapping:
   For each $t_i + \lambda_i K$, consider the point (t_i, λ_i) in $\mathbb{R}^{d+1}$.
2. Use the result of Danzer and Grünbaum in $\mathbb{R}^{d+1}$.

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Zs. Lángi, N, '09: If $K = -K$ then the maximum number of pairwise touching homothets of K is $< \frac{3}{2}2^d$.

Idea of the proof: The same mapping and something more...

Bezdek–Pach conjecture:

Maximum number of pairwise touching homothets of K is also 2^d.

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Bezdek–Pach conjecture:

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One-sided Hadwiger-number

$H^+(K)$: the maximum number of pairwise non-overlapping translates of K that touch K and whose translation vectors are in a closed half-space (with o at boundary).

Bezdek, Brass, '03

$H^+(K) \leq 2 \cdot 3^{d-1} - 1$, and equality exactly for parallelotopes.

Bezdek–Pach conjecture:

Maximum number of pairwise touching homothets of K is also 2^d.

Zs. Lángi, N, '09: If $K = -K$ then the maximum number of pairwise touching homothets of K is $< \frac{3}{2}2^d$.

An open one-sided Hadwiger-number-like quantity

$H_\infty^+(K)$: the maximum number of pairwise non-overlapping translates of K that contain o and whose translation vectors are in an open half-space (with o at boundary).

Zs. Lángi, N, '09

For $\hat{K} = -\hat{K} \subset \mathbb{R}^{d+1}$ we have $\bar{H}_\infty^+(\hat{K}) \leq 3 \cdot 2^{d-1}$ for the CLOSED one-sided Hadwiger-number-like quantity, and equality exactly for parallelotopes.

Bezdek–Pach Conjecture

An open one-sided Hadwiger-number-like quantity

$H_\infty^+(K)$: the maximum number of pairwise non-overlapping translates of K that contain o and whose translation vectors are in an open half-space (with o at boundary).

Zs. Lángi, N, '09

The following statements are equivalent.

1. There is a $K = -K \subset \mathbb{R}^d$ with n pairwise touching homothets.
2. There is a $\hat{K} = -\hat{K} \subset \mathbb{R}^{d+1}$ with $H_\infty^+(\hat{K}) \geq n$.

Thus, the problem is hard!

A Question by Füredi and Loeb '94

$K = -K$ convex body in $\mathbb{R}^d$ ($d > 2$). Is it true that the number of pairwise intersecting homothets of K which do not contain each other's centers is $\leq 2^d$?
Note: for the Euclidean disk in $\mathbb{R}^2$ it is ≥ 8.

Talata '05: False. Even for translates, it can be $> \frac{16}{35}\sqrt{7}^d$.

N, K. Swanepoel, J. Pach '15+:

$K = -K$ convex body in $\mathbb{R}^d$. Then the number of pairwise intersecting homothets of K which do not contain each other's centers in their interiors is $\leq e3^d(d+2)\ln d$.

For translates: $\leq 3^d$.

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For translates: $\leq 3^d$.
Puzzle: What is the maximum number of translates of a triangle on the plane that all contain the origin and none contains the centroid of the other in its interior?

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Idea of the proof (follows Füredi and Loeb)

The bound for translates is standard: use the isodiametric inequality (follows straight from the Brunn-Minkowski inequality).

For homothets: Apply a "logarithmic cut" of the homothety factors (ie., group them in intervals of the form $[(1+\varepsilon)^\ell, (1+\varepsilon)^{\ell+1}])$, and deal with the small ones as with translates.

Finally, deal with the large homothets by centrally projecting the centers onto the unit sphere. This step requires the use of a technical lemma (Bow and arrow inequality).

Lower bound — Symmetric case

Bourgain [in Füredi–Loeb paper]

For any number $s < \sqrt{2}$, there exists an $\varepsilon(s) > 0$, such that, in any normed space of dimension d, there is a $(1 + \varepsilon(s))^d$ element point set on the unit sphere with the property that the distances between distinct points are $\geq s$.

Thus, for any o-symmetric K, the number of pairwise intersecting translates not containing each other's center is exponentially large in d.

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Main tool:

Milman's Quotient of Subspace Theorem, '85:

$1 \leq k < d-1$, $\lambda = k/(d-1)$, $K = -K$ a convex body in $\mathbb{R}^d$. Then there are linear subspaces $E \leq F \leq \mathbb{R}^d$, and an ellipsoid $\mathcal{E}$ in E such that $\dim E = k$ and

$$\mathcal{E} \subseteq P_F(K) \cap E \subseteq c(\lambda)\mathcal{E},$$

where $c(\lambda)$ depends only on λ.

Lower Bound — Non-symmetric case

Non-symmetric Quotient of Subspace Theorem [Milman–Pajor,'00]:

$1 \leq k < d-1$, $\lambda = k/(d-1)$, K a convex body in $\mathbb{R}^d$ with the centroid at the origin. Then there are linear subspaces $E \leq F \leq \mathbb{R}^d$, and an ellipsoid $\mathcal{E}$ in E such that $\dim E = k$ and

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Centroid of Projection Lemma, [N, Swanepoel, Pach, '15+]

K a convex body in $\mathbb{R}^d$. Then there is a $(d-1)$-dimensional linear subspace $H \leq \mathbb{R}^d$ such that the centroid of $P_H(K)$ is the origin.

Sphere of Influence Graphs

$k \in \mathbb{Z}^+$, $\{c_i \colon i = 1, \ldots, m\}$ a set of points in $\mathbb{R}^d$ with norm $\|\cdot\|$.
$r_i^{(k)}$: the smallest r such that $\{j \in \mathbb{Z}^+ \colon j \neq i, \|c_i - c_j\| \leq r\}$ has at least k elements.

The k-th closed sphere-of-influence graph:

$V = \{c_i : \ i = 1, \ldots, m\}$

$\{c_i, c_j\}$ an edge if $B(c_i, r_i^{(k)}) \cap B(c_j, r_j^{(k)}) \neq \emptyset$.

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Applications: Image processing, pattern analysis.

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Guibas, Pach, Sharir, '91:

Maximum number of edges is $\leq nk(c^d - 1)$, for some constant $c > 1$.

N, K. Swanepoel, J. Pach '15+:

Maximum number of edges is $\leq nk(5^d - 1)$.

Happy 120, Egon and Károly!