# Sperner type lemmas and related topics

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# papers


O. R. Musin, Borsuk–Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* **140** (2012), 2551-2560.

O. R. Musin, Extensions of Sperner and Tucker's lemma for manifolds, *J. of Combin. Theory Ser. A,* **132** (2015), 172–187.

O. R. Musin, Sperner type lemma for quadrangulations, *Moscow J. of Combinatorics and Number Theory,* **5** (2015).

O. R. Musin & A. Yu. Volovikov, Borsuk–Ulam type spaces, *Mosc. Math. J.*, 2015+

O. R. Musin, Generalizations of Tucker–Fan–Shashkin lemmas, arXiv:1409.8637

O. R. Musin, Homotopy invariants of covers and KKM type lemmas, arXiv:1505.07629

# Brouwer fixed point theorem (1912)

**Brouwer fixed point theorem.**
*Any continuous map $f : B^n \to B^n$ must have a fixed point.*

Here $B^n$ denote an $n$-ball.

# Sperner lemma (1928)

## Theorem

**(Sperner lemma)** *Every Sperner labelling of a triangulation of a $d$-dimensional simplex contains a cell labelled with a complete set of labels:* $\{1, 2, \ldots, d+1\}$.

# Sperner lemma

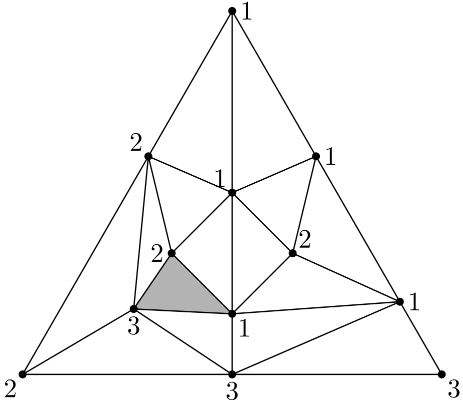



Figure: A 2-dimensional illustration of Sperner's lemma

# Degree and Sperner's lemma

**Theorem (M., 2014)**

*Let $T$ be a triangulation of a PL orientable $d$-dimensional manifold $M$ with boundary. Let $L : V(T) \to \{1, 2, \ldots, d+1\}$ be any labelling. Then $T$ contains at least $|\deg(f_L, \partial T)|$ fully labelled $d$-simplices, where $f_L : T \to \Delta^d$ and $\Delta^d$ is a $d$-dimensional simplex with vertices $1, 2, \ldots, d+1$.*

For Sperner's labelling $\deg(L, \partial T) := \deg(f_L, \partial T) = \pm 1$.

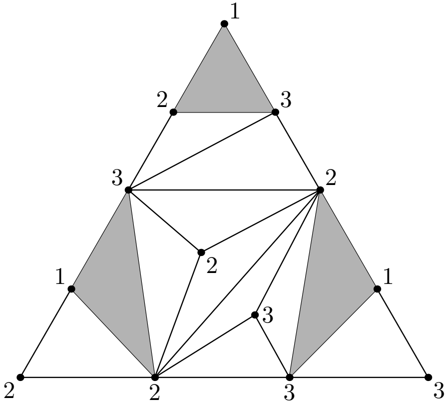



Figure: $\deg(L, \partial T) = 3$. There are three fully labelled triangles.

# A generalization of the De Loera - Petersen - Su theorem

**Theorem (M., 2014)**

*Let $P$ be a convex polytope in $\mathbb{R}^d$ with $n$ vertices. Let $T$ be a triangulation of a compact oriented PL–manifold $M$ of dimension $d$ with boundary. Let $L : V(T) \to \{1, 2, \ldots, n\}$ be a labelling such that $f_{L,P}(\partial M) \subseteq \partial P$. Then $T$ contains at least $(n - d)|\deg(L, \partial T)|$ fully labelled $d$-simplices.*

# A generalization of the De Loera - Petersen - Su theorem

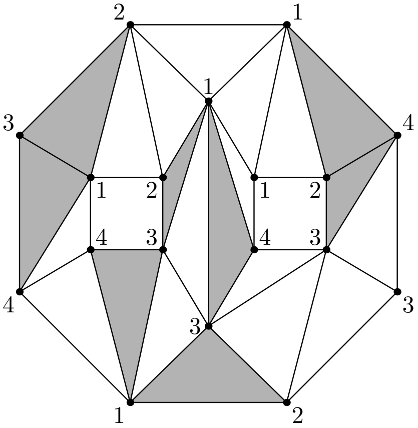



Figure: Octagon with two square holes. Here $n = 4$, $\deg(L, \partial T)| = 4$ and there are eight fully labelled triangles

# Sperner type lemma for quadrangulations

Denote by $C^d$ the $d$–dimensional cube.

**Theorem (M., 2015)**

*Let $Q$ be a quadrangulation of an oriented $d$-dimensional manifold $M$. Suppose $L : V(Q) \to V(C^d)$ be a labelling such that $f_L(\partial Q) \subseteq \partial C^d$. Then $Q$ contains at least $|\deg(L, \partial Q)|$ balanced labelled cells.*

# Sperner type lemma for quadrangulations

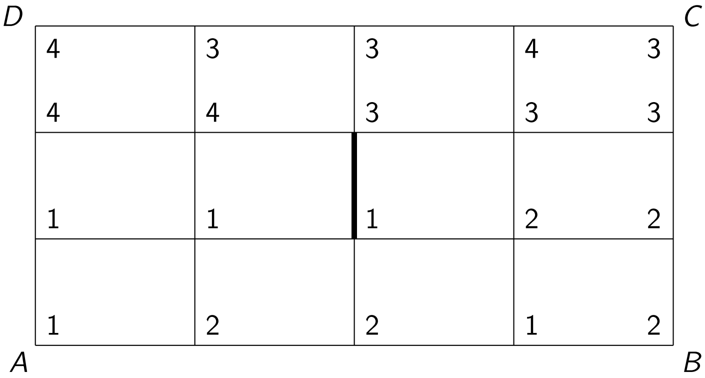



Fig.: Sperner's labelling of $\Pi_2(4,3)$. One edge is colored with $(1,3)$.

# Sperner type lemma for quadrangulations

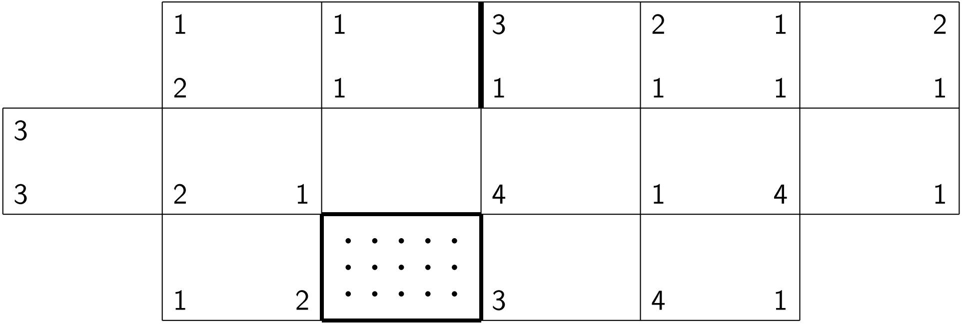



Fig.: Since $\deg(L, \partial Q) = 2$, there are two balanced labelled cells.

# The Borsuk-Ulam theorem (1933)

**The Borsuk - Ulam theorem (Borsuk, 1933).** Four equivalent statements:
(a) For every continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.
(b) For every antipodal (i.e. $f(-x) = -f(x)$) continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = 0$.
(c) There is no antipodal continuous mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.
(d) There is no continuous mapping $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ that is antipodal on the boundary.

# Tucker's lemma (1945)

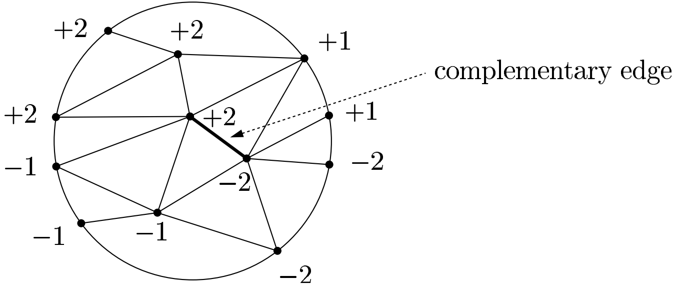
## Theorem (Tucker)

*Let $\Lambda$ be a triangulation of the ball $\mathbb{B}^d$ that is antipodally symmetric on the boundary. Let*

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be a labelling of the vertices of $\Lambda$ that satisfies $L(-v) = -L(v)$ for every vertex $v$ on the boundary $\mathbb{B}^d$. Then there exists an edge in $\Lambda$ that is "complementary": i.e., its two vertices are labelled by opposite numbers.*

# Tucker lemma



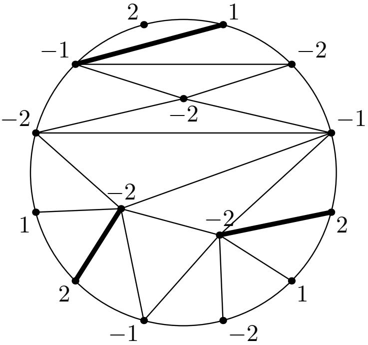



Figure: Since $\deg(L, \partial T) = 3$, there are three complementary edges.

# Tucker lemma for spheres

## Theorem

*Let $\Lambda$ be an antipodal triangulation of $\mathbb{S}^d$. Let*

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \dots, +d, -d\}$$

*be an antipodal labelling of the vertices of $\Lambda$ that satisfies $L(-v) = -L(v)$ for all vertices. Then $\Lambda$ contains a complimentary edge.*

# Shashkin lemma (1996)

## Theorem

*Let $T$ be a triangulation of a planar polygon that antipodally symmetric on the boundary. Let*

$$L : V(T) \to \{+1, -1, +2, -2, +3, -3\}$$

*be a labelling of the vertices of $T$ that satisfies $L(-v) = -L(v)$ for every vertex $v$ on the boundary. Suppose that this labelling does not have complementary edges. Then for any numbers $a, b, c$, where $|a| = 1$, $|b| = 2$, $|c| = 3$, the total number of triangles in $T$ with labels $(a, b, c)$ and $(-a, -b, -c)$ is odd.*

# Shashkin lemma

In other words, Shashkin proved that if $(a, b, c) = (1, 2, 3)$, $(1, -2, 3)$, $(1, 2, -3)$ and $(1, -2, -3)$, then the number of triangle with labels $(a, b, c)$ or $(-a, -b, -c)$ is odd. Denote this number by $SN(a, b, c)$. Then in the Figure we have
$SN(1, 2, 3) = 3,\ SN(1, -2, 3) = 1,$
$SN(1, 2, -3) = 3,\ SN(1, -2, -3) = 3.$
Note that, this result was published only in Russian and only for two–dimensional case. Moreover, Shashkin attributes this theorem to Ky Fan.

# Shashkin lemma

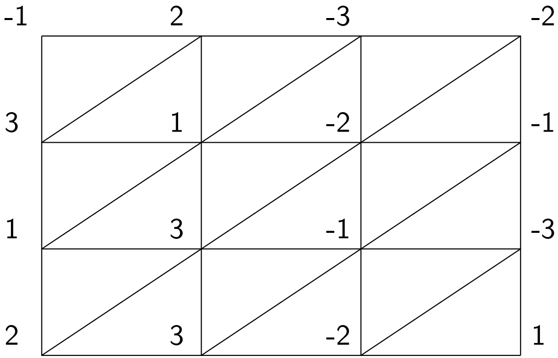



Figure 2: Illustration of Shashkin's lemma.

# Shashkin lemma

## Theorem

*Let $T$ be a centrally symmetric triangulation of $\mathbb{S}^d$. Let*

$$L : V(T) \to \Pi_{d+1} := \{+1, -1, +2, -2, \ldots, +(d+1), -(d+1)\}$$

*be an antipodal labelling of $T$. Suppose that this labelling does not have complementary edges. Then for any set of labels $\Lambda := \{\ell_1, \ell_2, \ldots, \ell_{d+1}\} \subset \Pi_{d+1}$ with $|\ell_i| = i$ for all $i$, the number of $d$–simplices in $T$ that are labelled by $\Lambda$ is odd.*

# Borsuk-Ulam theorem for manifolds

Our analysis of Bárány's proof shows that it can be extended for a wide class of manifolds. For instance, consider two-dimensional orientable manifolds $X = M_g^2$ of even genus $g$ and non-orientable manifolds $X = N_m^2$ with even $m$. We can assume that $X$ is "centrally symmetric" embedded to $\mathbb{R}^k$, where $k = 3$ for $X = M_g^2$ and $k = 4$ for $X = N_m^2$.

That means $A(X) = X$, where $A(x) = -x$ for $x \in \mathbb{R}^k$. Then $T := A|_X : X \to X$ is a free involution. It can be shown that there is a projection of $X \subset \mathbb{R}^k$ into a 2-plane $R$ passing through the origin 0 with $|Z_{f_0}| = 2$.

# $\mathbb{Z}_2$-maps

Let us consider a closed smooth manifold $M$ with a free smooth involution $T : M \to M$, i.e. $T^2(x) = x$ and $T(x) \neq x$ for all $x \in M$. For any $\mathbb{Z}_2$-manifold $(M, T)$ we say that a map $f : M^m \to \mathbb{R}^n$ is *antipodal* (or equivariant) if $f(T(x)) = -f(x)$.

We say that a closed $\mathbb{Z}_2$-manifold $(M, T)$ is a *BUT (Borsuk-Ulam Type) manifold* if for any continuous map $F : M^n \to \mathbb{R}^n$ there is a point $x \in M$ such that

$$F(T(x)) = F(x).$$

In other words, if a continuous map $f : M^n \to \mathbb{R}^n$ is antipodal, then the set $Z_f := f^{-1}(0)$ is not empty.

# BUT manifolds

**Theorem**

*Let $M^n$ be a closed connected manifold with a free involution $T$. Then the following statements are equivalent:*

*(a) For any antipodal continuous map $f : M^n \to \mathbb{R}^n$ the set $Z_f$ is not empty.*

*(b) $M$ admits an antipodal continuous transversal map $h : M^n \to \mathbb{R}^n$ with $|Z_h| = 4k + 2,\ k \in \mathbb{Z}$.*

*(c) $\mu(M, T) := (w_1^n(M/T), [M/T]) \neq 0$.*

*(d) $[M^n, T] = [\mathbb{S}^n, A] + [V^1][\mathbb{S}^{n-1}, A] + \ldots + [V^n][\mathbb{S}^0, A]$ in $\mathfrak{N}_n(\mathbb{Z}_2)$.*

# BUT–manifolds

(e) $M$ is a Lyusternik-Shnirelman type manifold, i.e. for any cover $F_1, \ldots, F_{n+1}$ of $M^n$ by $n+1$ closed (respectively, by $n+1$ open) sets, there is at least one set containing a pair $(x, T(x))$.

(f) $M$ is a Tucker type manifold, i.e. for any equivariant labelling $L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +n, -n\}$ of any equivariant triangulation $\Lambda$ of $M$ there exists a complementary edge.

(g) $M$ is a Ky Fan type manifold, i.e. for any equivariant labelling $L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +m, -m\}$ there is a complementary edge or an odd number of $n$-simplices whose labels are of the form $\{k_0, -k_1, k_2, \ldots, (-1)^d k_n\}$, where $1 \leq k_0 < k_1 < \ldots < k_n \leq m$.

# Shashkin lemma for BUT–manifolds

$$\Pi_{d+1} := \{+1, -1, +2, -2, \dots, +(d+1), -(d+1)\}$$

**Theorem**

*Let $(M, A)$ be a d-dimensional BUT–manifold. Let $T$ be an antipodally symmetric triangulation of $M$. Let $L : V(T) \to \Pi_{d+1}$ be an antipodal labelling of $T$. Suppose that this labelling does not have complementary edges. Then for any set of labels $\Lambda := \{\ell_1, \ell_2, \dots, \ell_{d+1}\} \subset \Pi_{d+1}$ with $|\ell_i| = i$ for all $i$, the number of d–simplices in $T$ that are labelled by $\Lambda$ is odd.*

# Tucker and Shashkin's lemma for manifolds with boundary

**Theorem**

*Let $M$ be a $d$–dimensional compact PL manifold with boundary $\partial M$. Suppose $(\partial M, A)$ is a BUT–manifold. Let $T$ be a triangulation of $M$ that antipodally symmetric on $\partial M$.*

**Tucker:** *Let $L : V(T) \to \Pi_d$ be a labelling of $T$ that is antipodal on the boundary. Then there is a complementary edge in $T$.*

**Shashkin:** *Let $L : V(T) \to \Pi_{d+1}$ be a labelling of $T$ that is antipodal on the boundary and has no complementary edges. Then for any set of labels $\Lambda := \{\ell_1, \ell_2, \dots, \ell_{d+1}\} \subset \Pi_{d+1}$ with $|\ell_i| = i$ for all $i$, the number of $d$–simplices in $T$ that are labelled by $\Lambda$ or $(-\Lambda)$ is odd.*

# Homotopy invariants of covers and Sperner type lemmas

With any labelling of a simplicial complex $T$ we associate certain homotopy classes of maps $T$ into spheres. These homotopy invariants can be considered as obstructions for extensions of covers of a subspace $A$ to a space $X$. We using these obstructions for generalizations of Sperner's lemma. In particular, we show that in the case when $A$ is a $k$–sphere and $X$ is a $(k+1)$–disk there exist Sperner type lemmas for covers by $n+2$ sets if and only if the homotopy group $\pi_k(\mathbb{S}^n) \neq 0$.

# Homotopy invariants of covers and Sperner type lemmas

Example 1: Let $T$ be a triangulation of a tetrahedron $S$ ($S = \mathbb{B}^3$). $L : V(\partial T) \to \{1,2,3\}$. If in $\partial T = \mathbb{S}^2$ there are no fully labeled triangles, then $f_L : \mathbb{S}^2 \to \mathbb{S}^1$. So $f_L$ is null–homotopic and $f_L$ can be extended to $f_L : \mathbb{B}^3 \to \mathbb{S}^1$.

# Homotopy invariants of covers and Sperner type lemmas

Example 2: Let $T$ be a triangulation of a simplex ($S = \mathbb{B}^{d+1}$). $L : V(\partial T) \to \{1, 2, 3, 4\}$. If in $\partial T = \mathbb{S}^d$ there are no fully labeled simplices, then we have $f_L : \mathbb{S}^d \to \mathbb{S}^2$. If $d \geq 2$, $\pi_d(\mathbb{S}^2) \neq 0$, then there are non null–homotopic maps (labelings). Thus, we have a Sperner type lemma.

For the case $d = 3$ we can construct a labelling from the Hopf fibration (map) $p : \mathbb{S}^3 \to \mathbb{S}^2$.

# Homotopy invariants of covers and Sperner type lemmas

*We say that a pair $(X, A)$ of spaces belongs $EP_n$ and write $(X, A) \in EP_n$ if $A$ is a subspace of a space $X$, there are non null–homotopic continuous maps $f : A \to \mathbb{S}^n$ and any $f$ with $[f] \neq 0$ in $[A, \mathbb{S}^n]$ cannot be extended to a continuous map $F : X \to \mathbb{S}^n$ with $F|_A = f$.*

We denoted this class of pairs by EP after S. Eilenberg and L. S. Pontryagin who initiated the obstruction theory in the late 1930s

# Homotopy invariants of covers and Sperner type lemmas

**Theorem**

*Let $(X, A) \in EP_{m-2}$ and let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a cover of $A$ such that the intersection of all $S_i$ is empty and $[\mathcal{S}] \neq 0$ in $[A, \mathbb{S}^{m-2}]$. If $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a cover of $X$ that extends $\mathcal{S}$, then all the $F_i$ have a common intersection point.*

# Homotopy invariants of covers and Sperner type lemmas

## Theorem

*Let* $X = |K|$ *and* $A = |Q|$, *where* $K$ *is a simplicial complex and* $Q$ *is a subcomplex of* $K$. *Suppose* $(X, A) \in EP_n$. *Let* $L : Vrt(K) \to \{1, 2, \ldots, m\}$ *be a labeling of* $K$. *Let* $V := \{v_1, \ldots, v_m\}$ *and* $p$ *be points in* $\mathbb{R}^{n+1}$. *Suppose there are no simplices in* $Q$ *whose vertices are labeled by* $J \in cov_V(p)$. *Let*

$$h(Q, L, V, p) \neq 0 \text{ in } [|Q|, \mathbb{S}^n].$$

*Then there are simplex* $s$ *in* $K$ *and* $J \in cov_V(p)$ *such that vertices of* $s$ *have labels* $J$.

*If* $m = n + 2$ *and* $[Q, L] \neq 0$ *in* $[|Q|, \mathbb{S}^n]$, *then there is a simplex in* $K$ *that has all labels* $1, \ldots, n + 2$.

Thank you