Kneser Transversals

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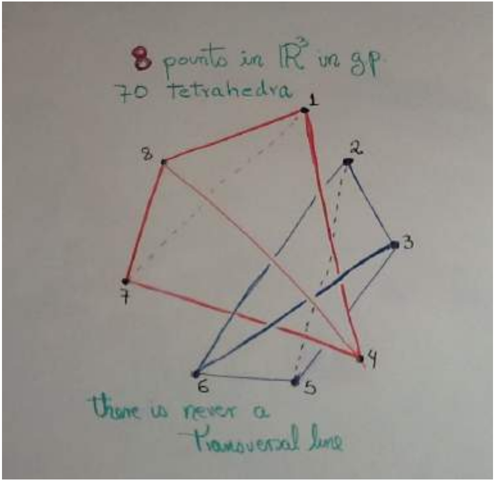
GEOMETRY AND SYMMETRY

Overview

8 points in R^3 in general position



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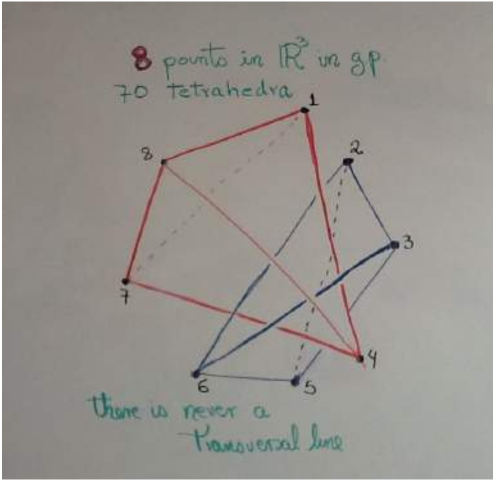
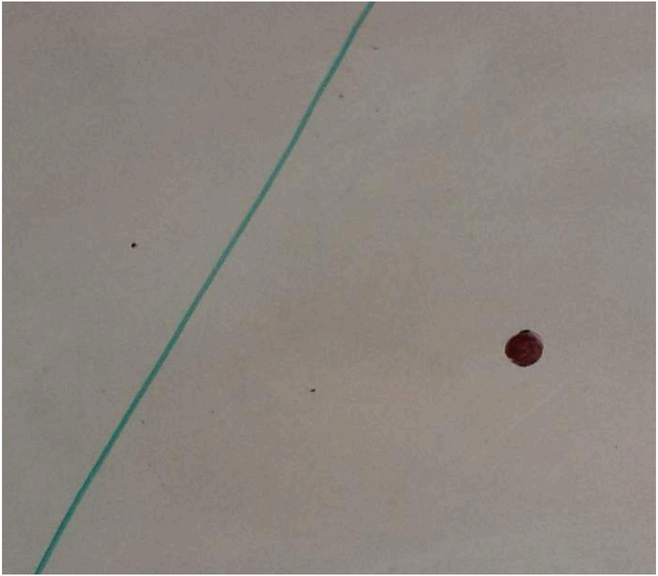


Is there a transversal line to the convex hull of all tethahedra ?

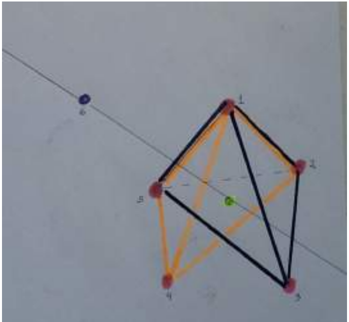
8 points in R^3 in general position NEVER



6 POINTS, ALWAYS

Given 6 points in $\{1, 2, 3, 4, 5, 6\} \subset R^3$ there is ALWAYS a transversal line to the convex hull of the tetrahedra with vertices in$\{1, 2, 3, 4, 5, 6\} \subset R^3$

{2,3,4,5}, {1,3,4,5}, {1,2,4,5}, {1,2,3,5},{1,2,3,4}
The thetrahedra with 6 as a vertex



7 POINTS, SOMETIMES YES, SOMETIMES NOT

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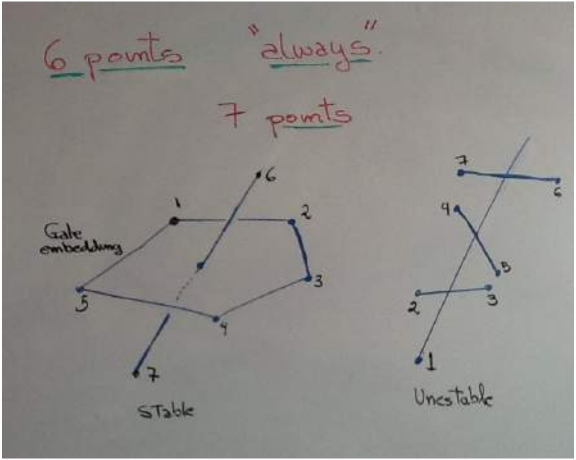

Basic Definitions

always

$m(k, d, \lambda) =$ the maximum positive integer n such that every set of n points in R^d has the property that the convex hull of all k-sets have a transversal $(d - \lambda)$-plane

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$M(k, d, \lambda) =$ the minimum positive integer n such that for every set of n points in $\mathbb{R}^d$ in general position, the convex hull of the k-sets do not have a transversal $(d - \lambda)$-plane.

$$m(k, d, \lambda) < M(k, d, \lambda)$$

$M(k, d, \lambda)$ NEVER

$$M(k, d, \lambda) = d + 2(k - \lambda) + 1$$

- $M(4, 3, 2) = 8$
- *The inequality:* $M(k, d, \lambda) \leq d + 2(k - \lambda) + 1$ *is a Combinatorial argument*
- *Gale Emmbeddings give rice to the inequality* $M(k, d, \lambda) \geq d + 2(k - \lambda) + 1$

$m(k, d, \lambda)$
ALWAYS

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda).$$

- $m(4, 3, 2) = 6$
- *The proof of this inequality uses Schubert calculus in the cohomology ring of Grassmannian manifolds.*
- *This inequality has strong connetions with the chromatic number of the Kneser hypergraphs*

$$6 \leq m(3,4,2)$$

Given 6 points in $\{1,2,3,4,5,6\} \subset R^4$ there is ALWAYS a transversal plane to the convex hull of the triangles with vertices in$\{1,2,3,4,5,6\}$.

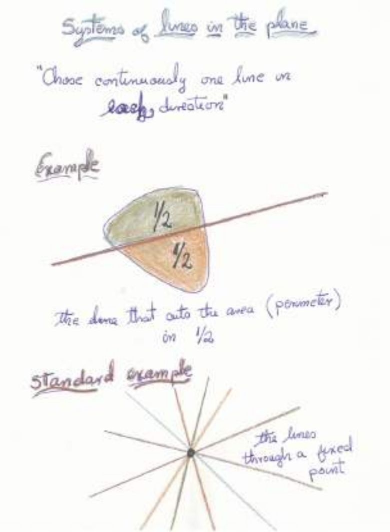
The triples of $\{1, 2, 3, 4, 5, 6\}$ can be colores with three color (red, blue and green) in such a way that the triples of every color satisfies the "2-Helly Property". That is "every 3 red triangles intersect", "every 3 blue triangles intersect", and "every 3 green triangles intersect".

{2,3,4}, {1,3,4}, {1,2,4}, {1,2,3},

The triangles with 5 as a vertex

The triangles with 6 as a vertex

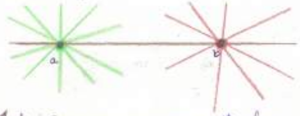
$$\chi(KG^3(6, 3)) \leq 3$$


Systems of lines in the plane
"Chose continuously one line in each direction"
Example
1/2
1/2
the line that cuts the area (perimeter) in 1/2
standard example
the lines through a fixed point

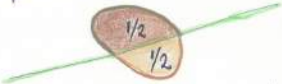
4

Fact "two systems of lines coincide in one direction"

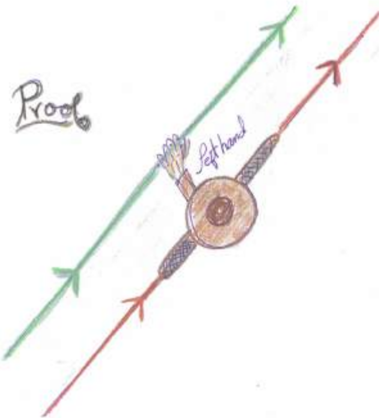
Obvious for two standard systems

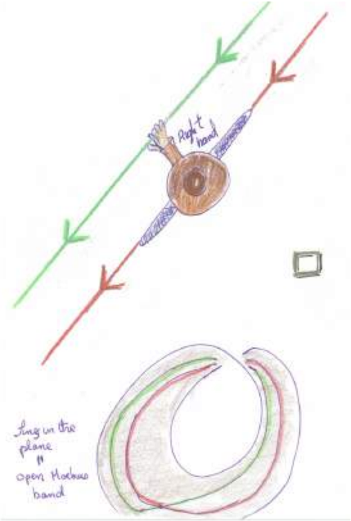


Fact Given a convex figure in the plane there is a line that cuts its area and its perimeter in 1/2



Fact Given a plane convex figure and a point A there is a line through A that cuts its area 1/2

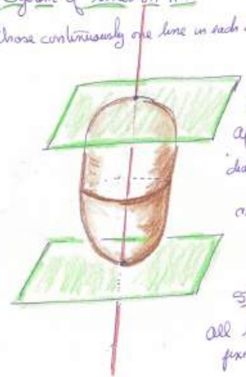

Proof
left hand


Right hand
Lines in the plane
open Moebius band

-7-

System of lines in $\mathbb{R}^d$

Chose continuously one line in each direction of $\mathbb{R}^d$



Example
affine diameters
or
diametral lines
of a strictly
convex body in $\mathbb{R}^d$

Standard example
all lines through a
fixed point.

Fact: two systems of lines in $\mathbb{R}^d$ coincide
in one direction.

Systems of planes in $\mathbb{R}^3$

Chose continuously one plane perpendicular to each direction

Standard example = planes through a fixed point

Example: planes that cuts the volume of a convex body in 1/2

Fact three system of planes coincide in one direction

Corollary given three points A, B, C there is a plane through A, B and C


the Ham Sandwich theorem
9
Given 3 bodies
there is a plane that
cuts the three
volumens simultaneoly
in
1/2

Systems of hyperplanes in $\mathbb{R}^d$

10

Chose continuously a hyperplane of $\mathbb{R}^d$ perpendicular to every direction

Fact: Given d systems of hyperplanes in $\mathbb{R}^d$ they coincide in one direction

Borsuk-Ulam.

Dolnikov.

there is not an antipodal map

$$S^{m+k} \longrightarrow S^m$$

Three systems of planes in R^4 coincide in one direction.

PROOF

$$6 \leq m(3, 4, 2)$$

- Green System of planes; All planes of R^4 through 5
- Blue System of planes; All planes of R^4 through 6
- There is a (red) plane IN EVERY DIRECTION transversal to the 4 red triangles

  Orthogonally projecting the 4 red triangles and then using Helly in the plane.

Kneser Graphs

- Let $[n]$ denote the set $\{1, ..., n\}$
- $\binom{[n]}{k}$ the collection of k-subsets of $[n]$
- The well known Kneser graph has vertex $\binom{[n]}{k}$ and two k-subsets are connected by an edge if they are disjoint.

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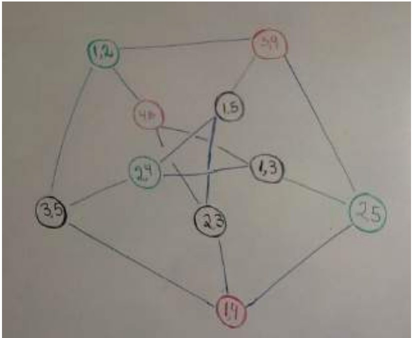
$$KG^2(5, 2)$$



Chromatic Number of Kneser Hypergraphs

The Kneser Hypergraph $KG^{\lambda}(n,k)$ has vertex $\binom{[n]}{k}$ and λ k-subsets $\{S_1, ..., S\lambda\}$ give rise to an hyperedge if $S_1 \cap ... \cap S_{\lambda} = \emptyset$.

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The chromatic number $\chi(KG^{\lambda+1}(n, k))$ of the Kneser hypergraph is the smallest number m such that a proper coloring of $KG^{\lambda}(n, k)$ with m colors exist.

Connection between $m(k, d, \lambda)$ and the Chromatic Number of Kneser Hypergraphs

The connections between the chromatic number and $m(k, d, \lambda)$ are of the following sort:

PROPER COLORATIONS

$\downarrow$

KNESER TRANSVERSAL

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The connections between the chromatic number and $m(k, d, \lambda)$ are of the following sort:

PROPER COLORATIONS

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KNESER TRANSVERSAL

If the triangles of a set X of 8 points in R^4 can be colored with 3 colors in such a way that the triangles of the same color has the 2-Helly property (3 by 3 have a common point), then there is a plane transversal to all triangles of X.

Upper bound for $m(k, d, \lambda)$

$$\downarrow$$

Lower bound for $\chi(KG^{\lambda+1}(n, k))$

Upper bound for $m(k, d, \lambda)$

$$\downarrow$$

Lower bound for $\chi(KG^{\lambda+1}(n, k))$

If $\quad m(k, d, \lambda) < n, \quad$ then $\quad d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k)).$

Upper bound for $m(k,d,\lambda)$

$\downarrow$

Lower bound for $\chi(KG^{\lambda+1}(n,k))$

If $m(k,d,\lambda) < n$, then $d-\lambda+1 < \chi(KG^{\lambda+1}(n,k))$.

As consequence, of the lower bound

$m(k,d,\lambda) < d - 2(k-\lambda) + 1 = M(k,d,\lambda)$

we obtain:

Upper bound for $m(k,d,\lambda)$

$\downarrow$

Lower bound for $\chi(KG^{\lambda+1}(n,k))$

If $m(k,d,\lambda) < n$, then $d-\lambda+1 < \chi(KG^{\lambda+1}(n,k))$.

As consequence, of the lower bound
$m(k,d,\lambda) < d-2(k-\lambda)+1 = M(k,d,\lambda)$
we obtain:

$$n-2k+\lambda \quad < \quad \chi(KG^{\lambda+1}(n,k))$$

.

As a Corollary we obtain a proof of the Kneser Conjecture first proved by L. Lovasz.

Lovasz

$$\chi(KG^2(n, k)) = n - 2k + 2$$

.

Rado's Central Point Theorem

Another interesting connection concerns the

Rado's Theorem

Let X be a finite set of n points in $\mathbb{R}^d$. Then there exist a point $x \in \mathbb{R}^d$ such that any closed half-space H through x contains at least $\left\lceil \frac{n}{d+1} \right\rceil$ points of X.

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Theorem

Let X be a finite set of n points in $\mathbb{R}^d$. Then there exist a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\left\lfloor \frac{n-d+2\lambda)}{\lambda+1} \right\rfloor + (d-\lambda)$ points of X.

Conjecture

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \quad = \quad m(k, d, \lambda)$$

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The inequality $m(k,d,\lambda) < M(k,d,\lambda)$ shows that the conjecture is true if either $\lambda = 1$, or $k \leq \lambda + 1$ or $k = 2, 3$.

Our next purpose is to improve the inequality

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k,d,\lambda) \quad < \quad d + 2(k - \lambda) + 1$$

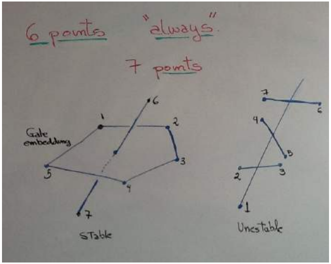
Next Purpose

Find an embedding of $d + 2(k - \lambda)$ *points in* R^d
WITHOUT
$(d - \lambda)$*-transversal line to the convex hull of their* k*-subsets*

Proving that $m(k, d, \lambda) < d + (k - \lambda)$.

In this range we have two classes of $(d - \lambda)$ Kneser Transversals

- Unestable
- Stable $=$ contain $(d - lambda + 1)$ of our points



We shall prove later that the cyclic politope with does not admit an stable Kneser Transversal

Special Kneser Transversals

Given a collection X of points in R^d a $(d - \lambda)$-plane L es called a special Kneser Transversal if and only if

- L contains $(d - lambda + 1)$ points of X
- L intersect the convex hull of all k-subsets of X

Oriented Matroids

order type $\rightarrow$ collection of of points

Kneser Transversals $\rightarrow$ Special Kneser Transversals

$$m(k, d, \lambda) \quad \rightarrow \quad m^*(k, d, \lambda)$$

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Geometry $\rightarrow$ Oriented Matroids

$$M^*(k, d, \lambda) = M(k, d, \lambda)$$

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If $2\lambda \geq d + 1$, then

$$m^*(k, d, \lambda) = d - \lambda + k$$

.

$$M^*(k,d,\lambda) = M(k,d,\lambda)$$

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.

$$k+(d-\lambda)+1 \leq m^*(k,d,\lambda) \leq \left\lfloor \left(\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil}\right)(k-1) \right\rfloor + (d-\lambda)+1.$$

Last inequality was obtained by proving that the cyclic polytope in $\mathbb{R}^d$ with more than

$$\left\lfloor \left(\frac{2\lceil\frac{d}{2}\rceil - \lambda + 1}{\lceil\frac{d}{2}\rceil}\right)(k-1) \right\rfloor + (d-\lambda) + 1.$$

vertices DOES NOT admit an special transversal $(d-\lambda)$-plane to the convex hulls of all their k-subset.

If $k - 1 > \lceil \frac{d}{2} \rceil$, then

$$\left\lfloor (\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil})(k-1) \right\rfloor + (d - \lambda) + 1 < 2d + (k - \lambda)$$

If $k-1 > \lceil \frac{d}{2} \rceil$, then

$$\left\lfloor \left(\frac{2\lceil \frac{d}{2} \rceil - \lambda + 1}{\lceil \frac{d}{2} \rceil}\right)(k-1) \right\rfloor + (d-\lambda) + 1 < 2d + (k-\lambda)$$

If $k-1 > \lceil \frac{d}{2} \rceil$, then

$$m(d,k,\lambda) < 2d + (k-\lambda)$$

Our Conjecture is true for $k = 2, 3, 4, 5 \quad \lambda = 1$ and $k \leq \lambda + 1$

The next interesting case is $d = 3, \lambda = 2$ and $k = 6$

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Martin Tancer was able to prove the Conjecture for $d = 3$, $k = 6$ and $\lambda = 2$. He consider the following special curve: the "moment curve" in the first two dimensions and a quickly growing function in the third dimension. He put 10 points in this curve and was able to prove that there is not a transversal line to the 6-gons.

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M. Tancer If $d = 3$ and $lambda = 2$

$$m(k, 3, 2) < 2k - 2 = M(k, 3, 2) - 2,$$

As a consequence of the some results of Matousek and Buck it is possible to prove, that our conjecture is not true at least in codimension 2.

If $d = 3$, $lambda = 2$ and $k \geq 7$

$$\left\lceil \frac{3k}{2} \right\rceil < \left\lceil \frac{5(k-1)}{3} \right\rceil \leq m(k, d, 2)$$

Coauthors

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- Jorge Ramirez Alfonsin
- Martin Tancer

THANK YOU

FOR YOUR

KIND ATTENTION

Karoly and Egon