Just a Snip

Barry Monson, University of New Brunswick

(from projects with L.Berman, D.Oliveros, and G.Williams)

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Where have we come from?

The abstract (= combinatorial) way of thinking about polyhedra and polytopes has old and deep, if somewhat disguised, roots:

- 17th-19th C and earlier: star-polyhedra of Kepler, Poinsot
- late 19th C: early work on regular maps
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- 1980's : Egon S.: Reguläre Inzidenzkomplexe
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So what are abstract polytopes?

An **abstract** n-**polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex n-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures.

You can safely think of a finite 3-polytope as a *map on a compact surface*.

Skip over the details?

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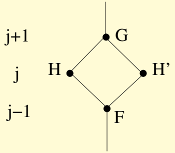
  whenever $F < G$ with $\text{rank}(F) = j-1$ and $\text{rank}(G) = j+1$ there exist exactly two j-faces H with $F < H < G$



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  via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra
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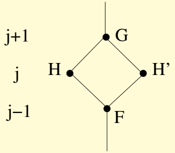

The symmetry of $\mathcal{Q}$

is described by its *automorphism group* $\mathrm{Aut}(\mathcal{Q})$.

(automorphism $=$ order-preserving bijection on $\mathcal{Q}$)

The axioms $\Rightarrow$ each automorphism is det'd by its action on any one *flag* Φ.

Example: for a polyhedron or 3-polytope $\mathcal{Q}$, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Defn. The *n*-polytope $\mathcal{Q}$ is *regular* if $\mathrm{Aut}(\mathcal{Q})$ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope

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Examples in rank $n = 3$: the convex regular polyhedra (=Platonic solids) and the Kepler-Poinsot star-polyhedra

Local data for both polyhedron $\mathcal{Q}$ and its group $\mathrm{Aut}(\mathcal{Q})$ reside in the Schläfli symbol or type $\{p, q\}$.

Platonic solids: $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron),$\{5, 3\}$ (dodecahedron)

Kepler (ca. 1619) $\{\frac{5}{2}, 5\}$ (small stellated dodecahedron), $\{\frac{5}{2}, 3\}$ (great stellated dodecahedron)

Poinsot (ca. 1809) $\{5, \frac{5}{2}\}$ (great dodecahedron), $\{3, \frac{5}{2}\}$ (great isosahedron)

Regular polytopes and string C-groups

Egon (1982 - almost a child) showed that the abstract regular n-polytopes $\mathcal{P}$ correspond exactly to the *string C-groups of rank n* (which we often study in their place).

How? having fixed a base flag Φ in $\mathcal{P}$, for $0 \leq j \leq n-1$ there is a unique automorphism $\rho_j \in \mathrm{Aut}(\mathcal{P})$ mapping Φ to the j-adjacent flag Φ^j.
The axioms $\Rightarrow$ these involutions generate $\mathrm{Aut}(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

and perhaps other relations, so long as this *intersection condition* continues to hold:

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle$$

(for all $I, J \subseteq \{0, \ldots, n-1\}$).

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$$\bullet \overset{p_1}{\text{———}} \bullet \overset{p_2}{\text{———}} \bullet \text{—} \ldots \text{—} \bullet \overset{p_{n-1}}{\text{———}} \bullet \, ,$$

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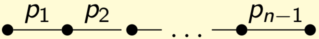

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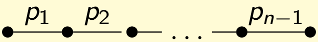
It's a hard life we lead ...

To repeat: $\mathrm{Aut}(\mathcal{P})$ is a quotient of the Coxeter group with diagram



We then say that the regular polytope $\mathcal{P}$ has Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

Those 'other' relations which induce this quotient can confound the intersection condition.

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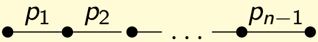

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Regular polytopes $\leftrightarrow$ string C-groups

The Correspondence Theorem (Egon, 1982)

Part 1. If $\mathcal{P}$ is a regular n-polytope, then $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a *string C-group*.

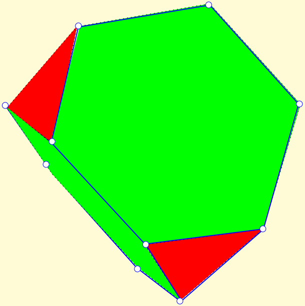
Part 2. Conversely, if $A = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an n-polytope $\mathcal{P}(A)$ (in a natural way as a coset geometry on A).

Furthermore, $\mathrm{Aut}(\mathcal{P}(A)) \simeq A$ and $\mathcal{P}(\mathrm{Aut}(\mathcal{P})) \simeq \mathcal{P}$.

Regularity is rare, despite its ubiquity

But most polytopes of rank $n \geq 3$ are not regular.

Eg. The truncated tetrahedron $\mathcal{Q}$, although quite symmetrical, has facets of two types (and 3 flag orbits under the action of $\mathrm{Aut}(\mathcal{Q}) \simeq S_4$).



Now lift to covers ...

- Likewise, a map $\mathcal{Q}$ on a compact surface will not usually be regular.
- But it is well-known that $\mathcal{Q}$ is covered by a regular map $\mathcal{P}$ (usually on some other surface).
- The regular cover $\mathcal{P}$ is unique (to isomorphism) if it covers $\mathcal{Q}$ minimally.
- The proof is straightforward and works for any abstract 3-polytope (e.g. if $\mathcal{Q}$ is a face-to-face tessellation of the plane). In fact,

$$\mathrm{Aut}(\mathcal{P}) \simeq \mathrm{Mon}(\mathcal{Q}), \text{ the monodromy group of } \mathcal{Q}\,.$$

- So it's crucial that $\mathrm{Mon}(\mathcal{Q})$ **is** a string C-group when rank $n = 3$.

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Monodromy scrambles the flags of an n-polytope $\mathcal{Q}$...

The *diamond condition* on $\mathcal{Q}$ amounts to this:

for each flag Φ and proper rank j $(0 \le j \le n-1)$ there exists a unique flag Φ^j which is j-adjacent to Φ (means ...)

So $r_j : \Phi \mapsto \Phi^j$ defines a fixed-point-free involution on the flag set $\mathcal{F}(\mathcal{Q})$.

Defn. The *monodromy group* $\mathrm{Mon}(\mathcal{Q}) := \langle r_0, \ldots, r_{n-1} \rangle$

(a subgroup of the symmetric group acting on $\mathcal{F}(\mathcal{Q})$).

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More on $\mathrm{Mon}(\mathcal{Q})$

- encodes combinatorial essence of $\mathcal{Q}$
- says much about how $\mathcal{Q}$ can be covered by an abstract regular n-polytope $\mathcal{P}$
- flag connectedness of $\mathcal{Q} \Rightarrow \mathrm{Mon}(\mathcal{Q})$ transitive on $\mathcal{F}(\mathcal{Q})$
- $\mathrm{Mon}(\mathcal{Q})$ is an sggi (= string group generated by involutions):

  r_j and r_k commute if $|j-k| > 1$
- $\mathrm{Mon}(\mathcal{Q})$ acts on $\mathcal{F}(\mathcal{Q})$ in a way contragredient to $\mathrm{Aut}(\mathcal{Q})$: for $g \in \mathrm{Mon}(\mathcal{Q})$, $\alpha \in \mathrm{Aut}(\mathcal{Q})$, flag $\Phi \in \mathcal{F}(\mathcal{Q})$

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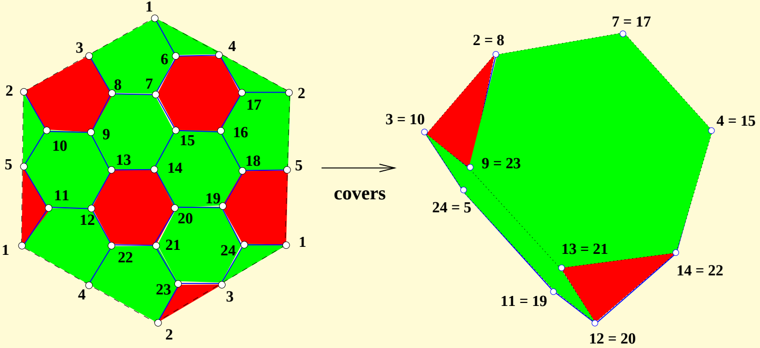
Let's get back to an example.

Hartley and Williams (2009) determined the minimal regular cover $\mathcal{P}$ for each classical (convex) Archimedean solid $\mathcal{Q}$ in $\mathbb{E}^3$.

Here the regular toroidal map $\mathcal{P} = \{6,3\}_{(2,2)}$ covers the truncated tetrahedron $\mathcal{Q}$.



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found

Theorem For $n \geq 2$, let $M_n = \langle r_0, r_1, \ldots, r_{n-1} \rangle$ be the monodromy group of the truncated n-simplex. Then

(a) M_n is a string C-group of type $\{6, 3, \ldots, 3\}$.

(b) M_n is isomorphic to $S_{n+1} \times S_n$.

(c) A presentation for M_n comes from adjoining to the standard relations for the Coxeter group with diagram $\bullet \overset{6}{\text{—}} \bullet \text{—} \bullet - \cdots - \bullet \text{—} \bullet$ (on n nodes) just one extra magic relation:

$$(r_0 r_1 r_0 r_1 r_2)^4 = e.$$

(for $n \geq 3$). This relation is independent of rank.

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(a) For $n \geq 1$, the truncated n-simplex has an essentially unique minimal regular cover $\mathcal{P}_n$ with $n!\,(n+1)!$ flags.

(b) For $n \geq 4$, $\mathcal{P}_n$ is the universal regular polytope for facets of type $\mathcal{P}_{n-1}$ and simplicial vertex-figures.

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In fact ...

The regular polytope $\mathcal{P}_n$ with $\mathrm{Aut}(\mathcal{P}) \simeq S_{n+1} \times S_n$ and Schläfli type $\{6, 3, \ldots, 3\}$ has surfaced before several times:

(a) its dual $\mathcal{P}_n^*$ was described by Egon as a regular incidence-polytope in 1985.

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(c) it is the *symmetric graphicahedron* based on the complete bipartite graph $K_{1,n}$ (= the n-star)
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Yet more regular polytopes, many of them new...

$\mathrm{Aut}(\mathcal{P}_n) \simeq S_{n+1} \times S_n$ is a rich supply of subgroups which are themselves string C-groups (see our paper in *J. Algebr. Comb.*, 2015):

j	Type	Order	Comments
0	$\{6, 3, \ldots, 3\}$	$(n+1)!n!$	$\mathrm{Aut}(\mathcal{P}_n)$ to start
1	$\{3, 6, 3, \ldots, 3\}$	$(n+1)!(n-1)!$	
j	$\{3, \ldots, 3, 6, 3, \ldots, 3\}$	$(n+1)!(n-j)!$	
$n-2$	$\{3, \ldots, 3, 6\}$	$(n+1)!2!$	
$n-1$	$\{3, \ldots, 3, 3\}$	$(n+1)!$	S_{n+1}, n-simplex

Project: reinterpret this construction using CPR graphs.

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And a more subtle class of related regular polytopes

Again $\mathrm{Aut}(\mathcal{P}_n) \simeq S_{n+1} \times S_n$. For $0 \leq j \leq n-1$, there is a subgroup $W_n(j)$ such that

- $W_n(j)$ has index $\binom{n}{j+1}$ in $\mathrm{Aut}(\mathcal{P}_n)$
- $W_n(j) \simeq S_{n+1} \times S_{j+1} \times S_{n-j-1}$
- $W_n(j)$ is a string C-group of type $\{3, \ldots, 3, 6, 3, 6, 3, \ldots, 3\}$ (first '6' in the jth position)
- $W_n(j)$ and $W_n(n-j-2)$ are isomorphic in dual fashion.

Note: actually $W_n(0)$ has type $\{3, 6, 3, 3, \ldots, 3\}$

This relates to ...

a more general fact concerning certain *Coxeter groups*:

For $p \geq 2$, the Coxeter group of rank n and diagram

$$\bullet\overset{2p}{\text{———}}\bullet\overset{3}{\text{———}}\bullet - \cdots - \bullet\overset{3}{\text{———}}\bullet$$

has a subgroup of index $\binom{n}{j+1}$ which is isomorphic in turn to the Coxeter group with diagram

$$\bullet\overset{3}{\text{———}}\bullet - \cdots - \bullet\overset{3}{\text{———}}\bullet\overset{2p}{\text{———}}\bullet\overset{p}{\text{———}}\bullet\overset{2p}{\text{———}}\bullet\overset{3}{\text{———}}\bullet - \cdots - \bullet\overset{3}{\text{———}}\bullet\ ,$$

where the first "$2p$" labels the j th branch of the diagram.

Is it known

when and in what circumstances a subset of n *reflections* in a Coxeter group

$$W = \langle r_0, \ldots, r_n \rangle$$

(of rank n), themselves generate a Coxeter group of rank n?

Note: **Defn**: here, a *reflection* in W is any conjugate of some r_j. This makes sense in the context of the standard linear representation of W.

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References

[1] L. Berman, B. Monson, D. Oliveros and G. Williams, *The monodromy group of a truncated simplex*, J. Algebr. Comb., 2015.

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