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Classification of fundamental domains for cocompact plane groups

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H. Poincaré (1882) attempted to describe a plane crystallographic group in the Bolyai-Lobachevsky hyperbolic plane $\mathbb{H}^2$ by appropriate fundamental polygon. This initiative he extended also to space. B. N. Delone (Delaunay) in 1960's refreshed this very hard topic for Euclidean space groups by the so-called stereohedron problem: To give all fundamental domains for a given space group; with few partial results.

A. M. Macbeath (1967) completed the initiative of H. Poincaré in classifying the 2-orbifolds by giving each with a signature. That is by a base surface with orientable or non-orientable genus; by some singular points on it, as rotational centers with given periods; by some boundary components, in each with given dihedral corners. All these are characterized up to equivariant isomorphism, also reported in this talk.

Abstract – 2

There is a nice curvature formula that describes whether the above (good) orbifold, i.e. cocompact plane group (with compact fundamental domain) is realizable either in the sphere $\mathbb{S}^2$, or in the Euclidean plane $\mathbb{E}^2$, or in the hyperbolic plane $\mathbb{H}^2$, respectively.

Our initiative in 1990's was to combine the two above descriptions; namely, how to give all the combinatorially different fundamental domains for any above plane group. Z. Lučić and E. Molnár completed this by a graph-theoretical tree enumeration algorithm. That time N. Vasiljević implemented this algorithm to computer (program COMCLASS), of super-exponential complexity, by certain new ideas as well. In the time of the Yugoslav war we lost our manuscript, then the new one has been surprisingly rejected (?!).

Now we have refreshed our manuscript to submit again. Here you are presented a report on it.

References – 1

Z. Lučić, E. Molnár and N. Vasiljević,
Combinatorial algorithm for computer classification of fundamental polygons for a cocompact plane discontinuous group of given signature,
manuscript

[4] B.N. Delone, N.P. Dolbilin and M.I. Shtogrin, Combinatorial and metrical theory of planigons, *Tr. Mat. Inst. Steklov Akad. Nauk SSSR* 148 (1978), 109–140 (in Russian); English translation: *Proc. Steklov Inst. Math.* 4 (1980), 111-141.

[6] D. Huson, The generation and classification of tile-k-transitive tilings of the euclidean plane, the sphere and the hyperbolic plane, *Geom. Dedicata*, 47 (1993), 269–296.

[7] Z. Lučić and E. Molnár, Combinatorial classification of fundamental domains of finite area for planar discontinuous isometry groups, *Arch. Math*, 54 (1990), 511–520.

[8] Z. Lučić and E. Molnár, Fundamental domains for planar discontinuous groups and uniform tilings, *Geom. Dedicata*, 40 (1991), 125–143.

[9] A.M. Macbeath, The Classification of non-Euclidean Plane Crystallographic Groups, *Can. J. Math.*, 19 (1967), 1192–1205.

References – 2

[12] E. Molnár, On D-Symbols and Orbifolds in an Algorithmic Way. *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia*, **58** (2011), 263–276.

[13] E. Molnár, E. Stettner, Symmetry groups and fundamental tilings for the compact surface of genus 3− (2. The normalizer diagram with classification), *Beiträge zur Algebra und Geometrie (Contr. Alg. Geom.)*, 46/1 (2005) 19–42.

[14] J.M. Montesinos, *Classical Tessellations and Three-Manifolds*, Springer, Berlin Heidelberg New York London Paris Tokyo, 1987.

[15] H. Poincaré, Théorie des groupes fuchsiens, *Acta Mathematica*, 1(1):1–62, December 1882.

I thank my son, Zsolt Molnár for preparing this presentation

Introduction – 1

Suppose that $\widetilde{M}$ is 2-dimensional, closed, compact manifold, with occasional singular points and boundaries, i.e. $\widetilde{M} = \Pi/G$ will be a good orbifold, as a compact fundamental domain F of an isometry group G acting discontinuously on a classical plane Π of constant curvature. The situation will be described step-by-step in the following. We can consider $\widetilde{M} = F$ as a polygon with side pairing identifications, i.e. with piecewise linear (PL) presentation on the affine plane A^2 (see [19]). By Macbeath's signature (see [9], [10], [18] or [19])

$$(g, \pm; [h_1, \ldots, h_l]; \{(h_{11}, \ldots, h_{1l_1}), \ldots, (h_{q1}, \ldots, h_{ql_q})\}),$$

where $\pm$ is $+$ or $-$, and where g, h_i $(1 \leq i \leq l)$ and h_{ij} $(1 \leq i \leq q, 1 \leq j \leq l_i)$ are integers such that $g \geq 0$, $h_i \geq 2$ and $h_{ij} \geq 2$, we express that:

(i) If $\pm = +$ and $g > 0$, then $\widetilde{M}$ is orientable surface of genus g, which means that $\widetilde{M}$ is a connected sum of g tori (Figure 1.a); if $g = 0$, then $\widetilde{M}$ is homeomorphic to a sphere; if $\pm = -$ and $g > 0$, then $\widetilde{M}$ is non-orientable surface of genus g, which means that $\widetilde{M}$ is a connected sum of g projective planes.

Introduction – 2

(ii) There are l *singular points* on $\widetilde{M}$, with *periods* $h_1, h_2, \dots, h_l$. If the set of periods is empty, i.e. if [ ] appears in a signature, then $\widetilde{M}$ has no singular points.

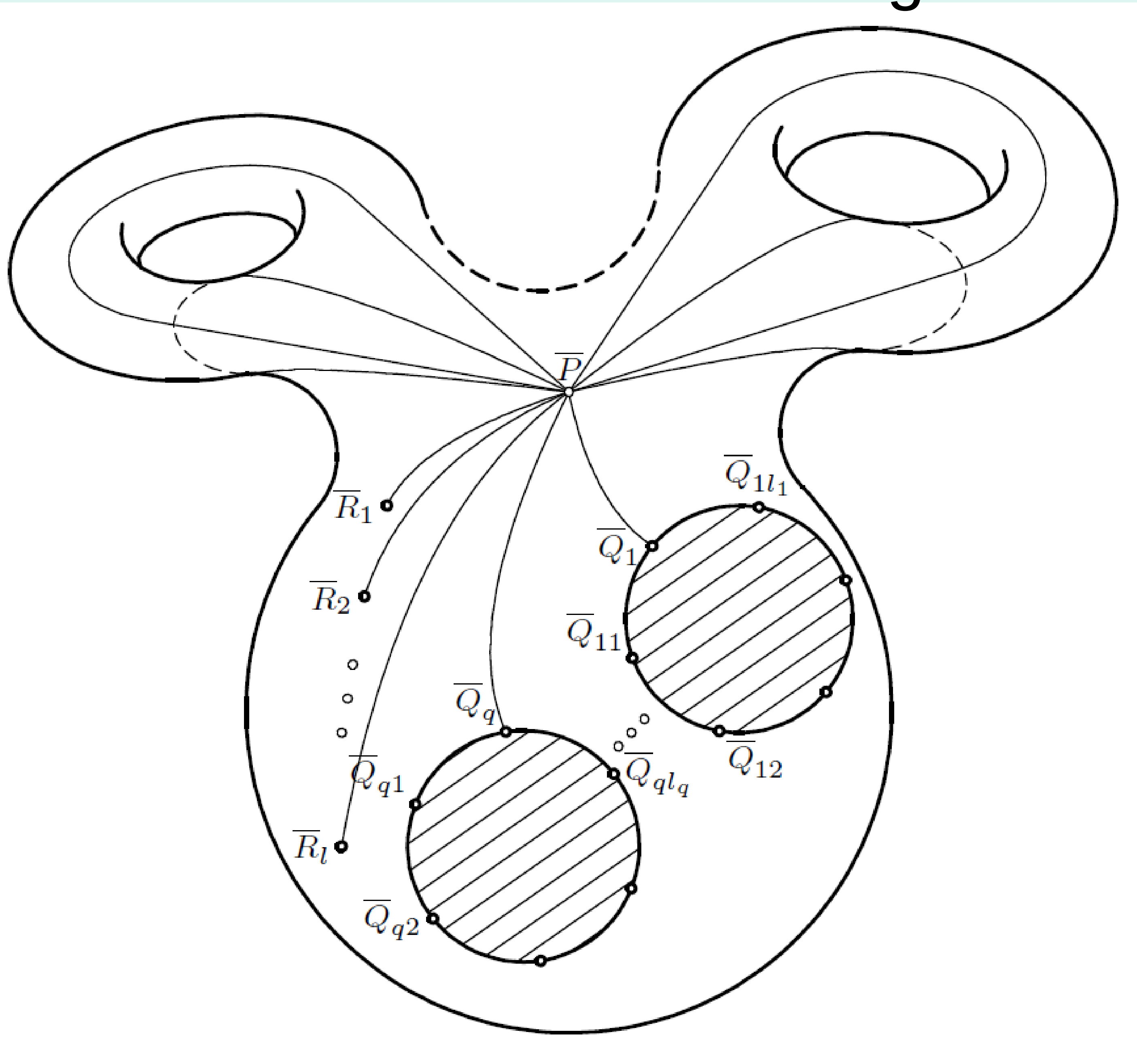
(iii) There are q disjoint closed (Jordan) curves $\gamma_1, \gamma_2, \dots, \gamma_q$ (called *boundary components*) on $\widetilde{M}$ and l_i $(1 \leq i \leq q)$ *dihedral points* on the curve γ_i, of *periods* h_{ij} $(1 \leq i \leq q, 1 \leq j \leq l_i)$. If $q = 0$ then $\{\}$ denotes that there is no boundary component.

Equivalent to the Macbeath's signature is Conway's orbifold notation (see [3] or [1]):

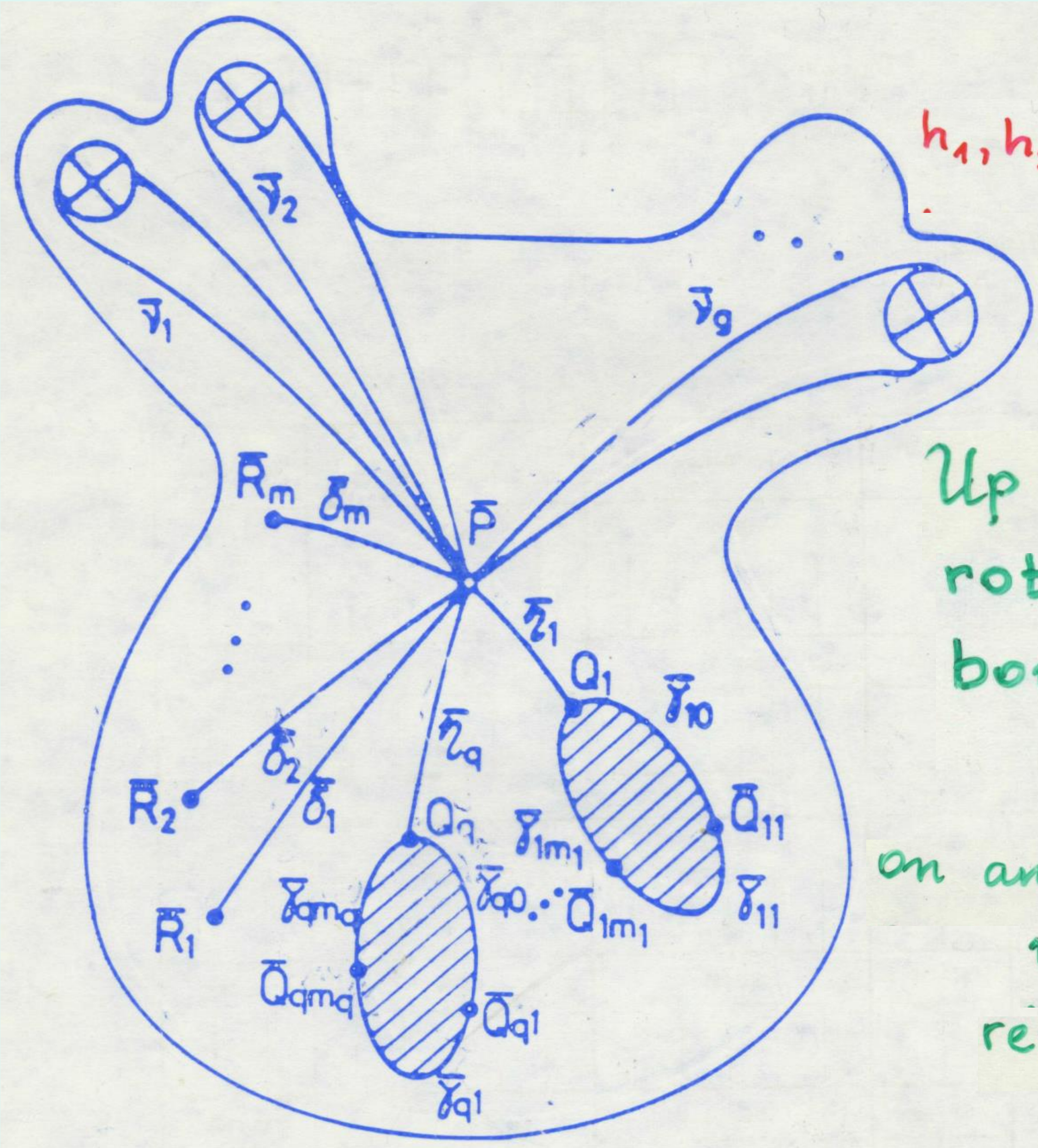
$$\circ \circ \dots \circ \quad h_1, \dots, h_l \quad * h_{11}, \dots, h_{1l_1} \; \dots \; * h_{q1}, \dots, h_{ql_q} \quad \times \times \dots \times,$$

either with g initial circles which represent tori (for orientable case), or g final crosses which represent projective planes (for non-orientable case). The absence of circles and crosses obviously indicates that the base manifold M of the orbifold $\widetilde{M}$ is a sphere.

The orientable orbifold – Fig.1.a



The non-orientable orbifold - Fig. 1.a



Signature:

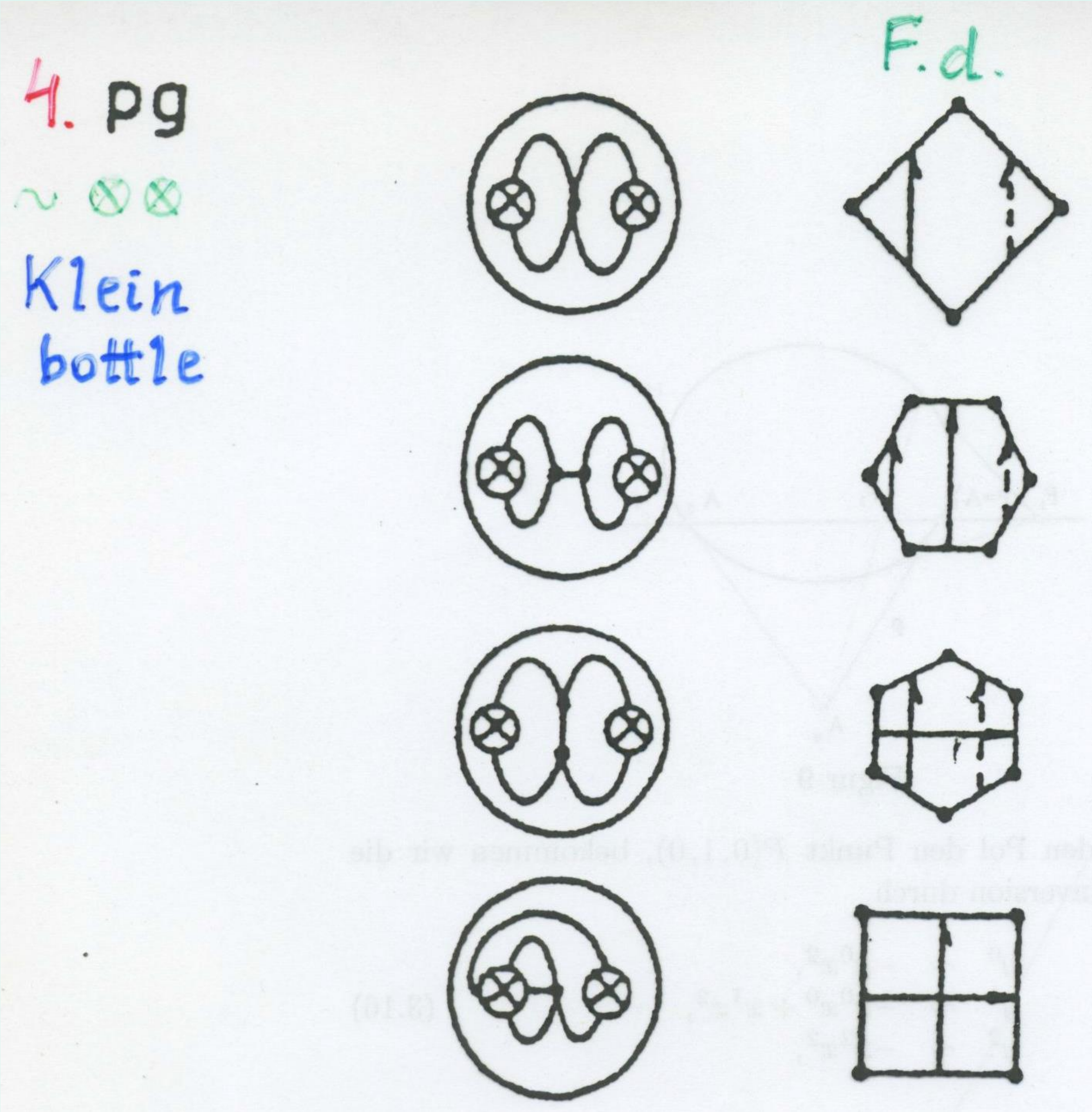
$$h_1, h_2, \ldots, h_m * h_{11}, h_{12}, \ldots, h_{1m_1} \ldots * h_{q1}, h_{q2}, \ldots, h_{qm_q} \otimes^g$$

Up to permutation of rotational orders, boundary components,

respectively

on any boundary component: up to cyclic and reverse cyclic permutation of dihedral corners.

Orbifold trees and their fundamental domains – 1



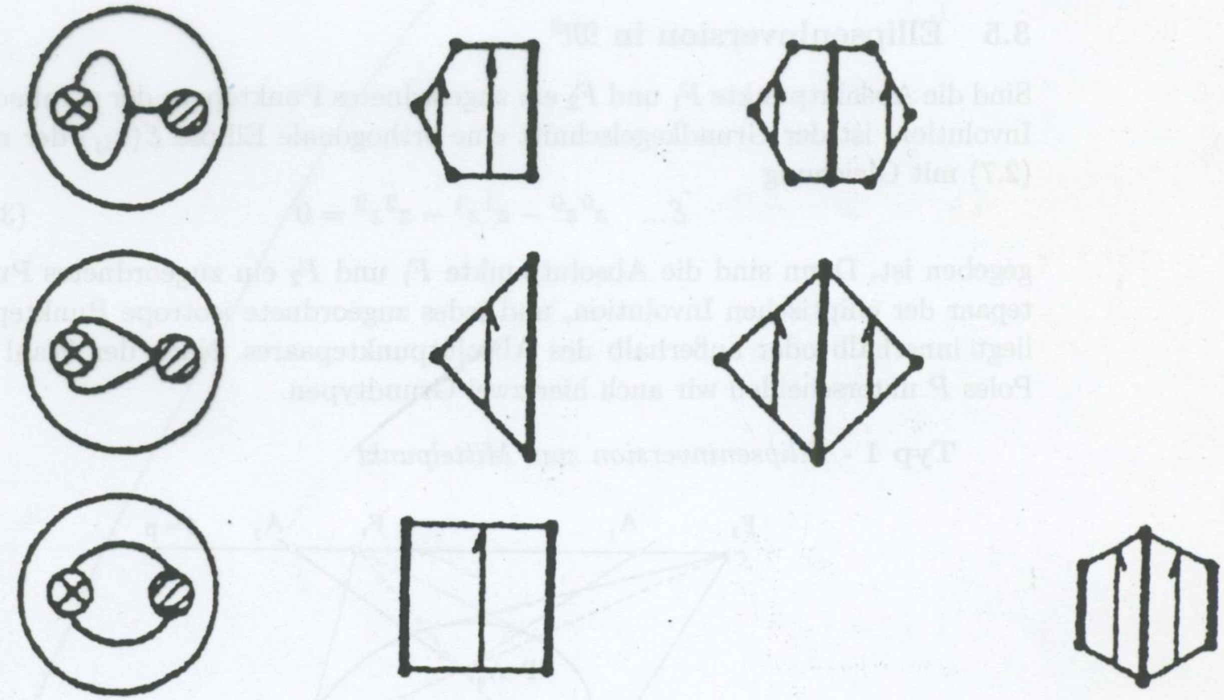
No multiple tile

Orbifold trees and their fundamental domains – 2

5. cm

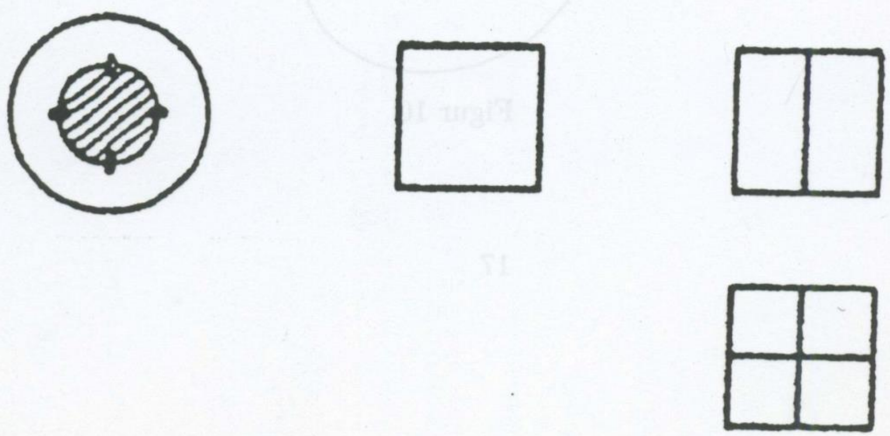
~ * ⊗

Möbius band



6. pmm

~ *2,2,2,2



multiple tiles

And vice versa

If G is finitely generated isometry group, with a fundamental domain $F = \Pi/G = \widetilde{M}$, which acts discontinuously on a complete, simply connected 2-dimensional manifold Π of constant curvature 0, +1 or −1 (i. e. Π is the Euclidean plane $\mathbb{E}^2$, or the 2-sphere $\mathbb{S}^2$, or the Bolyai-Lobachevskian hyperbolic plane $\mathbb{H}^2$), then a Macbeath's signature may serve as a *signature of* G in order to indicate the orientability ($\pm$) of Π/G, its genus (g), the orders $h_1, \ldots, h_l$ of the rotation centers and the stabilizers of the orders $2h_{ij}$ $(1 \le i \le q, 1 \le j \le l_i)$ associated with the dihedral centers on the i-th boundary component.

Identifying points from the same orbit of G, by a covering map

$$\kappa : \Pi \to \Pi/G, \qquad X \mapsto \overline{X} := X^G,$$

we obtain a surface $\widetilde{M} = \Pi/G$ which is a *good orbifold* [14, p. 87] (compact surface) if all the rotation and dihedral centers of G are of finite order.

At most points of Π the above map κ is a local homeomorphism. This is not at points with non-trivial stabilizers, hence, at the rotation centers and at the points mapped onto the boundary of $\widetilde{M}$.

We give the well known necessary and sufficient conditions for topological (homeomorphically equivariant) or geometrical isomorphism of planar discontinuous groups, where a topological mapping ϕ of Π induces the group isomorphism $\Upsilon : G \to G'$, $g \mapsto g' = \phi^{-1} g \phi$ (for the proof see [9] or [19, th. 4.6.3-4]).

Equivariance of groups

Two plane discontinuous groups G and G' are topologically isomorphic (equivariant) if and only if:

(a) The surfaces $\widetilde{M} = \Pi/G$ and $\widetilde{M}' = \Pi/G'$ are homeomorphic.

(b) The numbers of the non-equivalent rotation centers are the same and the orders of the rotations are the same, i.e. up to their permutation.

(c) On each boundary curve γ_i of $\widetilde{M}$ and γ_i' of $\widetilde{M}'$, respectively, there is a cycle of dihedral centers with corresponding orders $2h_{i1}, \ldots, 2h_{il_i}$. If $\widetilde{M}$ and $\widetilde{M}'$ are orientable, then either both have the same cycles or all those of $\widetilde{M}'$ are inverse to those of $\widetilde{M}$. If $\widetilde{M}$ and $\widetilde{M}'$ are non-orientable, then the cycles of $\widetilde{M}$ may be put in bijective correspondence with those of $\widetilde{M}'$, where image and pre-image may have the same or opposite orientation.

By a logical (formal) contraction of the q boundary disks into q singular points of the **compact** surface $\widetilde{M}$, we obtain a compact surface M^* without boundary, with q additional singular points and with the same rotation centers, genus and orientability as the starting $\widetilde{M}$.

Fundamental polygon F

For a given plane discontinuous group G there is a simply connected closed set, called a *fundamental domain of* G, whose G-images cover Π without any interior point in common (see [19, p. 115]). Moreover, for a fundamental domain of G may serve a *generalized polygon*, i.e. a topological disk F whose boundary is divided by a finite set of *vertices* into piecewise linear sides. This polygon we call a *fundamental polygon*. The sides of F are identified (or κ-paired) by isometries of Π which geometrically generate G. Generalized polygon F which serves as a fundamental domain of G, together with the set of identifications defined by G, is said to be *κ-paired polygon* of G. The vertices of F (where at least three G-images of F meet) fall into G-equivalence classes with G-conjugate stabilizers such that the indicator function takes the same value on them.

If Y is a midpoint of an edge such that $S(G_Y) = 2^+$, this point is (exceptionally) considered as a vertex of F, although only two G-images of F meets around Y. This is the point where our method differs from D-symbol method given in [1] and [6].

If a line reflection appears as a generator in G, a side on that line is unidentified and, because of that, this side appears on a boundary cycle of $\widetilde{M}$.

In order to characterize polygons which serve as fundamental domains for a given discontinuous group we give the following statement proved in [8] (see Fig. 1.a, b, c, d).

Fig.1.b

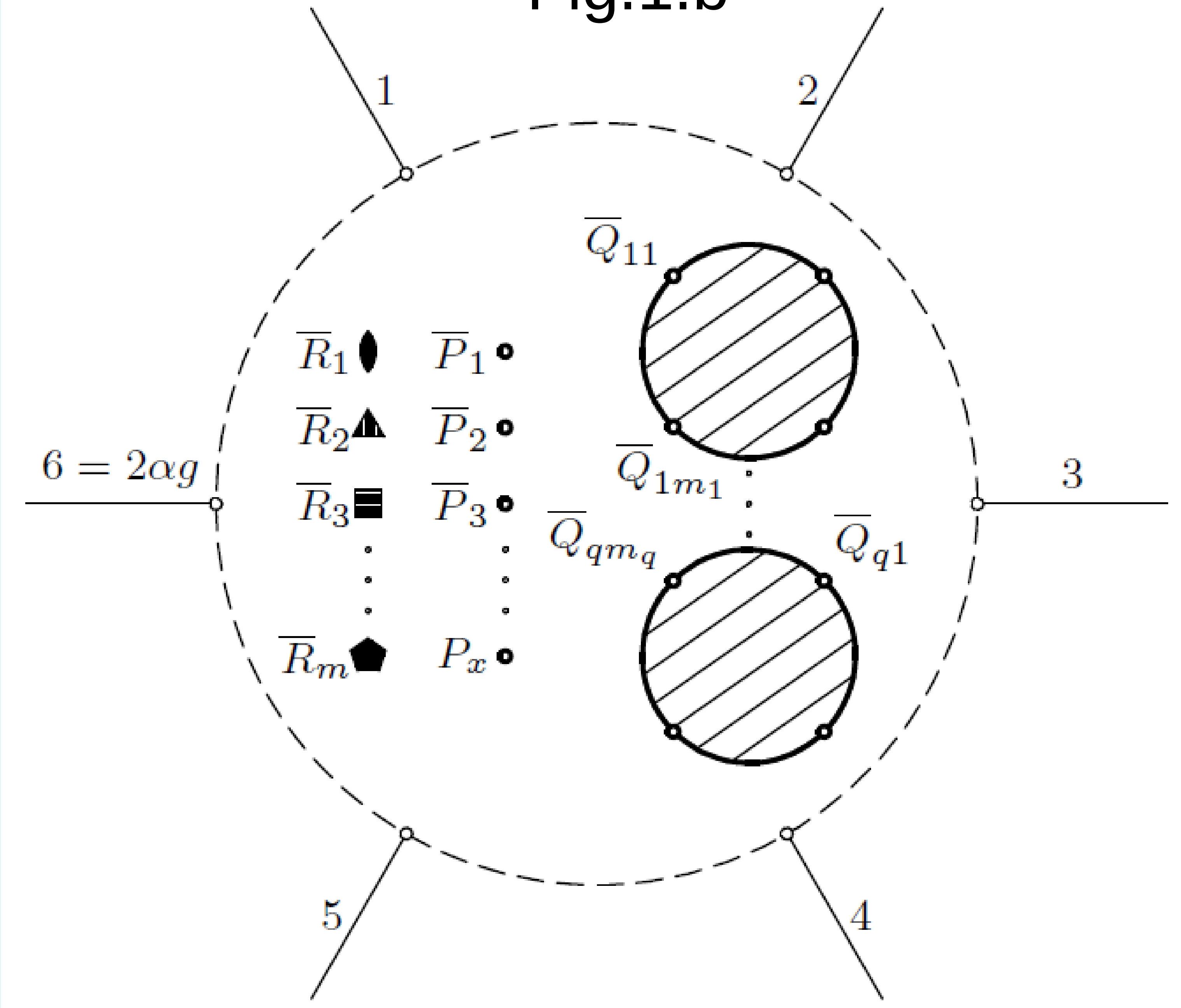
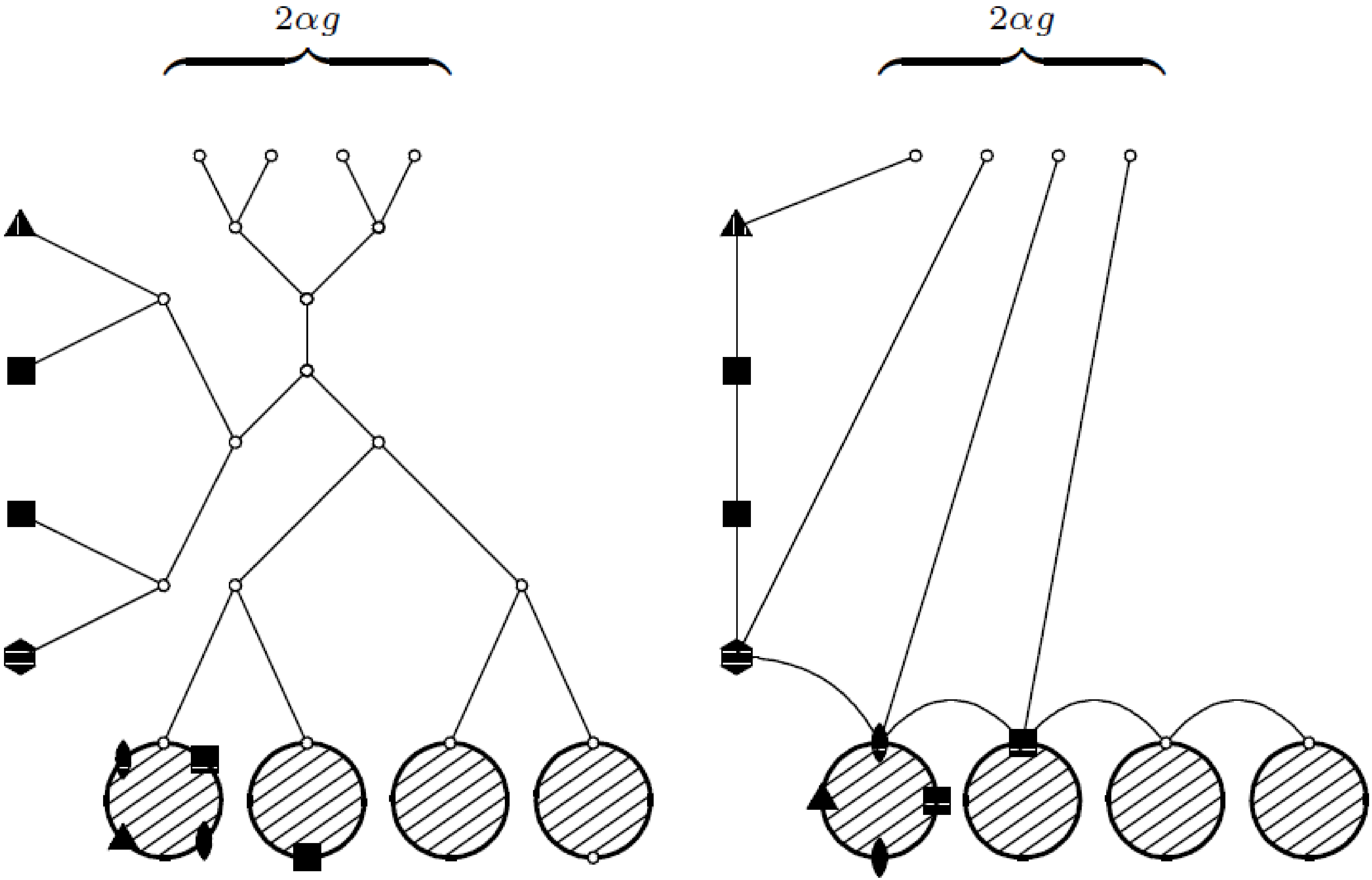

1
2
3
4
5
$6 = 2\alpha g$
$\overline{Q}_{11}$
$\overline{Q}_{1m_1}$
$\overline{Q}_{qm_q}$
$\overline{Q}_{q1}$
$\overline{R}_1$
$\overline{R}_2$
$\overline{R}_3$
$\overline{R}_m$
$\overline{P}_1$
$\overline{P}_2$
$\overline{P}_3$
P_x

Fig.1.c,d



Theorem 1

Theorem 1 *A κ-paired polygon F serves as a fundamental domain for a planar discontinuous group G given by a Macbeath's signature, if and only if κ-images of its sides form a graph C on a surface $\widetilde{M} = \Pi/G$ with the following properties:*

1. *$\widetilde{M} \setminus C$ is an open disk.*

2. *Graph C can be contracted on the surface $\widetilde{M}$ with genus g into the graph $\widetilde{C}$ with one vertex of valence $\nu = 2\alpha g$, and αg loops ($\alpha = 2$ if M is orientable, $\alpha = 1$ otherwise).*

3. *Each singular point $\overline{R}_i$, which is a κ-image of a rotation center R_i, is a vertex of C with valence $\nu(\overline{R}_i) \geq 1$.*

4. *Each subgraph C_i of C which belongs to the boundary component $\overline{b}_i$ can be contracted (as $\widetilde{M} \to M^*$ logical contraction indicated above) to the point $\overline{Q}_i$ with valence $\nu(\overline{Q}_i) \geq 1$.*

5. *Each vertex $\overline{P}$ of C, which is a κ-image of a vertex P of F with trivial stabilizer, has a valence at least 3.* □

Two fundamental (or κ-paired) polygons are said to be *combinatorially equivalent* if there is a bijection mapping one onto the other which preserves the relation of incidence of vertices and edges, their cyclic order, and the G-equivalence of vertices and the directed edges together with their stabilizers (see [7, p. 511]).

Theorem 2

Theorem 2 *Suppose that G is a finitely generated discontinuous isometry group with a comact fundamental domain, acting on Π. Suppose further that G is given with a fixed good Macbeath's signature, different from 4 types of bad orbifolds [14, p. 87]*

$$(0,+;[u];\{\ \}), \quad (0,+;[\ ];\{(u)\}), \quad 2 \leq u,$$
$$(0,+;[u,v];\{\ \}), \quad (0,+;[\ ];\{(u,v)\}, \quad 2 \leq u < v,$$

when G does not exist, and different from three types

$$S_1 = (0,+;[\ ];\{\ \}), \quad S_{\overline{1}} = (1,-;[\ ];\{\ \}),$$

$$R = (0,+;[\ ];\{(h_{11},\ldots,h_{1l_1})\}), \quad 0 < l_1,$$

with combinatorially unique fundamental domains. The set of all the combinatorially different polygons, which serve as fundamental domains for G, are obtained by the following procedure:

(a) *On $\widetilde{M}$ we determine the finite set (up to combinatorial equivalence) of all the possible non-contractible graphs with one vertex and αg loops ($\alpha = 2$ if $\widetilde{M}$ is orientable, $\alpha = 1$ otherwise), such that for any $\widetilde{C}$ from that set, $\widetilde{M} \backslash \widetilde{C}$ is a disk.*

Theorem 2 – cont.

(b) *To the graph* $\widetilde{C}$ *we let correspond a disconnected graph* $\widetilde{C}'$ *which consists of* αg *disjoint paths belonging to the loops of* $\widetilde{C}$ *(*$\widetilde{C}'$ *can be obtained by cutting a small disk* D *on* $\widetilde{M}$ *around the added vertex, Figure 1.b).*

(c) *We determine the finite set of all possible trees on* $\widetilde{M}$ *(in* D*), each of them meets* $\widetilde{C}'$ *only at the set of its* $2\alpha g$ *vertices (on the boundary of* D*), such that the set of vertices of each of these trees consists of:*

- (i) $2\alpha g$ *vertices of* $\widetilde{C}'$*, each is of valence one.*
- (ii) l *rotation centers* $\overline{R}_1, \ldots, \overline{R}_l$.
- (iii) *Points* $\overline{Q}_1, \ldots, \overline{Q}_q$ *obtained by contractions of the boundary components of* M.
- (iv) *Some additional points* $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_x$ *on* $\widetilde{M}$*, each is of valence at least three, whence* $x \leq 2\alpha g + l + q - 2$.

(d) *We join each of these trees with* $\widetilde{C}'$ *and replace* $\overline{Q}_1, \ldots, \overline{Q}_q$ *by the boundary components* $\overline{b}_1, \ldots, \overline{b}_q$ *of* M *with dihedral centers on them as new vertices, to obtain a new graph* C *on* M.

(e) *To every disk* $M \setminus C$ *we let correspond a polygon* F *which serves as a fundamental domain for* G.

(f) *Among all the polygons* F *we select the combinatorially different ones.* $\square$

Curvature formula

The inequality $x \leq 2\alpha g + l + q - 2$ (that was omitted in [8] but already mentioned in [7]) is justified by the simple fact from graph theory that, for a tree, the number of vertices of degree at least 3 is at least two less than the number of vertices of degree 1, which is in our case the sum of $2\alpha g$, and the number of vertices of degree 1 among $\overline{R}_1, \dots, \overline{R}_l$, and $\overline{Q}_1, \dots, \overline{Q}_q$, which number is at most $l + q$.

By comparing the angle sum of the polygon which serves as a fundamental domain of the planar discontinuous group G, given by Macbeath's signature, with the angle sum of the corresponding Euclidean polygon, we conclude that the group G is realizable as a group of isometries which acts discontinuously on $\mathbb{S}^2(<)$, in $\mathbb{E}^2(=)$ or $\mathbb{H}^2(>)$ if and only if

$$0 \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} 4 - 2\alpha g - 2\sum_{i=1}^{l}(1 - 1/h_i) - 2q - \sum_{j=1}^{q}\sum_{k=1}^{l_j}(1 - 1/h_{jk}), \quad \begin{matrix} \mathbb{H}^2 \\ \mathbb{E}^2 \\ \mathbb{S}^2 \end{matrix}$$

where $\alpha = 2$ if $\widetilde{M}$ is orientable, $\alpha = 1$ otherwise.

We give sharp estimates for the number $\mathfrak{n}$ of sides of a fundamental polygon F obtained by the procedure described in Theorem 2 (see [7]):

Theorem 3

Theorem 3 *If* $\mathfrak{n}$ *is the number of edges (and vertices) of an arbitrary fundamental polygon of finite area for a discontinuous group* G *given by its Macbeath's signature, different from the groups* S_1, $S_{\overline{1}}$ *and* R *with combinatorially unique fundamental domains, then*

$$\mathfrak{n}_{\min} \le \mathfrak{n} \le \mathfrak{n}_{\max}$$

where

$$\mathfrak{n}_{\min} = 2\alpha g \quad \text{if} \quad l = q = 0,$$

or

$$\mathfrak{n}_{\min} = q_0 + \sum_{k=1}^{q} l_k + 2\alpha g + 2l + 2q - 2$$

otherwise, and

$$\boxed{\mathfrak{n}_{\max} = \sum_{k=1}^{q} l_k + 6\alpha g + 4l + 5q - 6,}$$

where $\alpha = 2$ *if* $\widetilde{M}$ *is orientable and* $\alpha = 1$ *otherwise, and* q_0 *is the number of the boundary components of* M *containing no dihedral center. Moreover, for a given* G *there exist fundamental domains with* $\mathfrak{n}_{\min}$ *and* $\mathfrak{n}_{\max}$ *edges.* □

Theorem 4

Theorem 4 *For any given plane discontinuous group G of compact fundamental domain, given by its signature and described in Theorems 2 and 3, there are finitely many combinatorially different fundamental polygons. There exists an algorithm that enumerates all fundamental polygons for a given signature of G.*

$\square$

Remark:

We could think that the above procedure—based on the enumeration of graph theoretical trees belonging to given $2\alpha g+l+q$ fixed vertices and $x \leq 2\alpha g+l+q-2$ additional vertices, each of valence at least 3—can be extended for infinite order of rotation center and dihedral center, respectively.

Namely, in the Macbeath signature we could allow $h_i = \infty$ $(1 \leq i \leq l)$ as extended rotation center, an end in $\mathbb{H}^2$ for a horocyclic rotation, or an ideal point of two parallel line in $\mathbb{E}^2$ for an Euclidean translation. Furthermore, we could allow $h_{ij} = \infty$ $(1 \leq i \leq q, 1 \leq j \leq l_i)$ as extended dihedral center with parallel reflection lines in $\mathbb{H}^2$ or an ideal point of two parallel reflection line in the Euclidean plane $\mathbb{E}^2$. Moreover, to a boundary component could belong more dihedral centers at the absolute of $\mathbb{H}^2$ which will be contracted into a point in our procedure.

As we see these extended centers cause some difference in the above procedure to determine fundamental polygons for the above group G. The difference also appears in the metric realization of the corresponding fundamental domain with ideal vertices.

Theorem 5

Theorem 5 *[8, Prop. 3.2] Among all convex polygons in $\mathbb{S}^2$ or in $\mathbb{H}^2$ (resp. in $\mathbb{E}^2$) with given angles $\alpha_1, \alpha_2, \ldots, \alpha_m, m \geq 3$, there exists up to an isometry (resp. similarity) respecting the order of angles, exactly one circumscribing a circle.* $\square$

In Sec. 3 we will describe a particular algoritm which existence is stated in Theorem 4, and which is based on the procedure from Theorem 2 and facts from Theorem 3.

The computer implementation of that algoritm was done by the third author in his B.Sc. thesis [17]. The product of this implementation is program COMCLASS (see section 4). The complexity of the procedure, which is clearly super-exponential (observe that the number of labeled trees on n vertices is n^{n-2}, see [2]), will be discussed elsewhere (see e.g. [16]).

Particular problems have independently been solved in [1], [6], [10] and [16], partially by different methods. This gives us an opportunity to illustrate some of the steps in the procedure only by examples and figures.

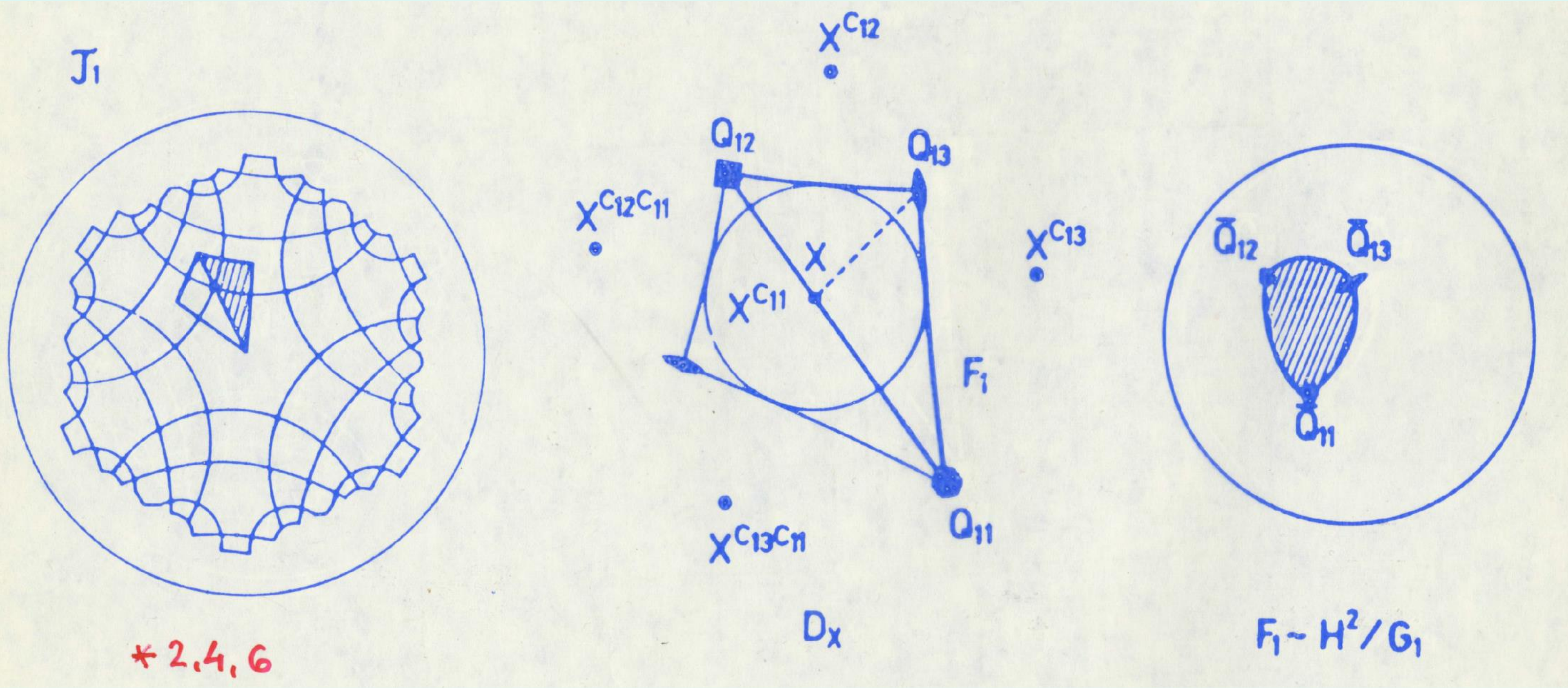
The Poincaré-Delone (Delaunay) problem

Thus, a long standing problem of H. Poincaré and B.N. Delone (Delaunay) has been completely solved in dimensions 2 by the theorems 2 – 5 , and the algorithm described in the Section 3. Our procedure also completes the classification of plane discontinuous groups with fundamental domains of finite area, finalized in [18], [9], [19]. Or, the corresponding 2-orbifolds have been completely described [14].

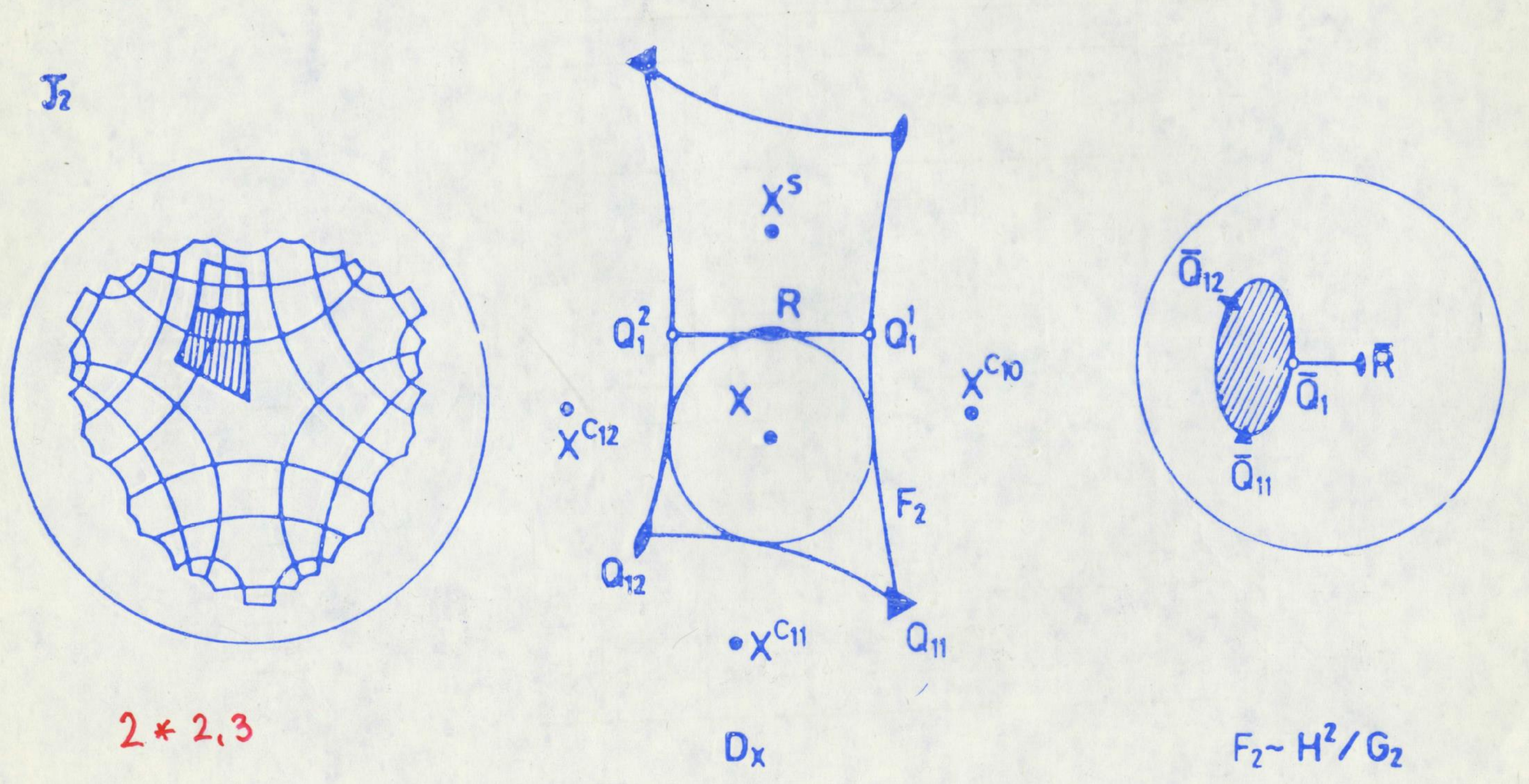
Poincaré [15] initiated finding hyperbolic plane groups via finding their possible fundamental domains. Delone (see e.g. [4]) looked for the general stereohedron problem, i.e. finding "all" combinatorial polyhedra (polytopes) which can be fundamental domains of some discrete groups acting discontinuously on a space of constant curvature (only particular cases are solved for dimensions greater than 2).

Our program is available in source code for on-line execution (see section 4). Its response is a list of all combinatorially different fundamental domains (represented by canonized polygon descriptors list) for a group given by its Macbeath's signature.

1. Different Archimedian tilings with the same Schläfli-symbol (4, 4, 4, 6): $\mathcal{T}_1 \neq \mathcal{T}_2$

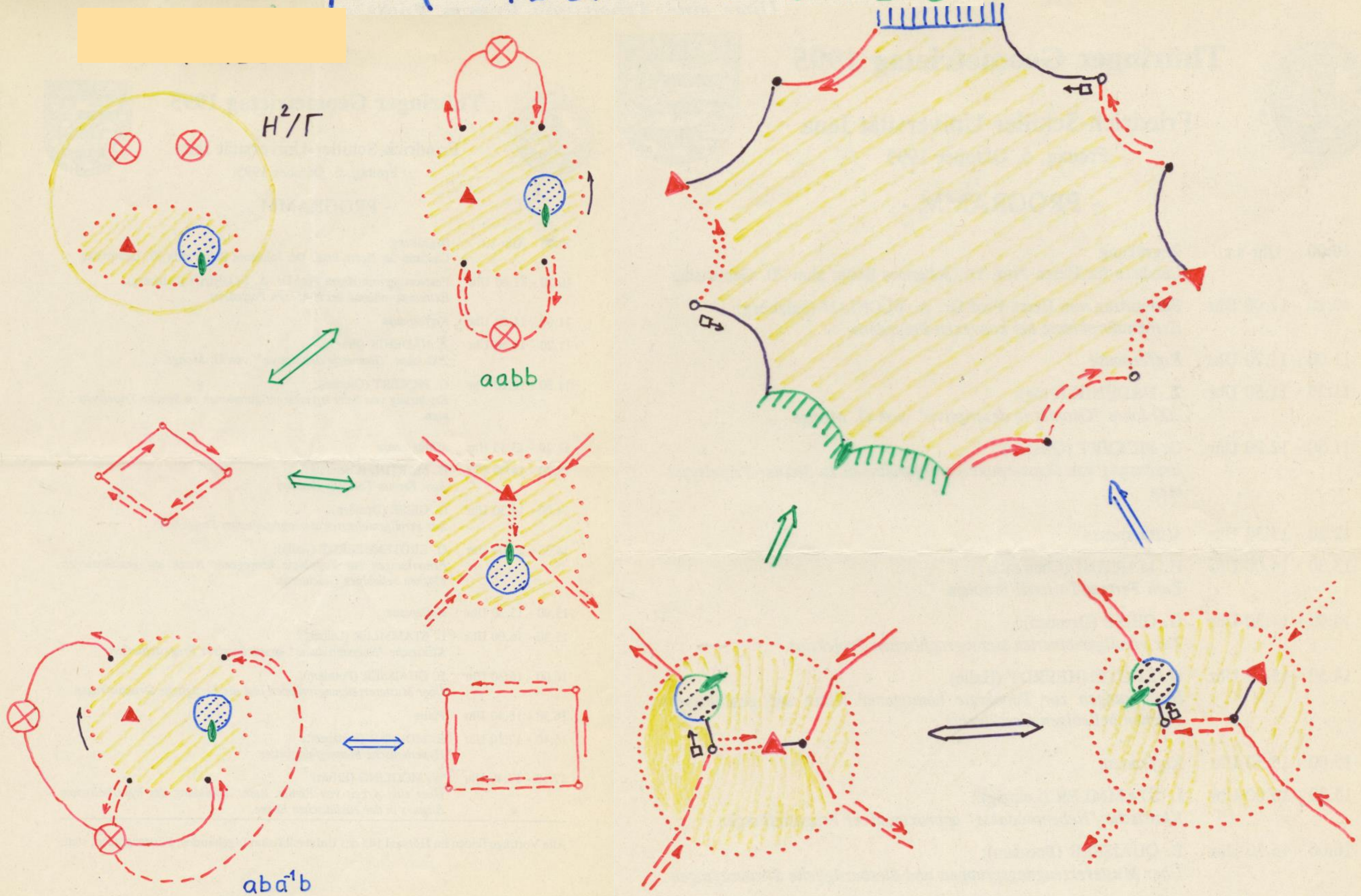


2. Different Archimedian tilings with the same Schläfli-symbol (4, 4, 4, 6): $\mathcal{T}_1 \neq \mathcal{T}_2$



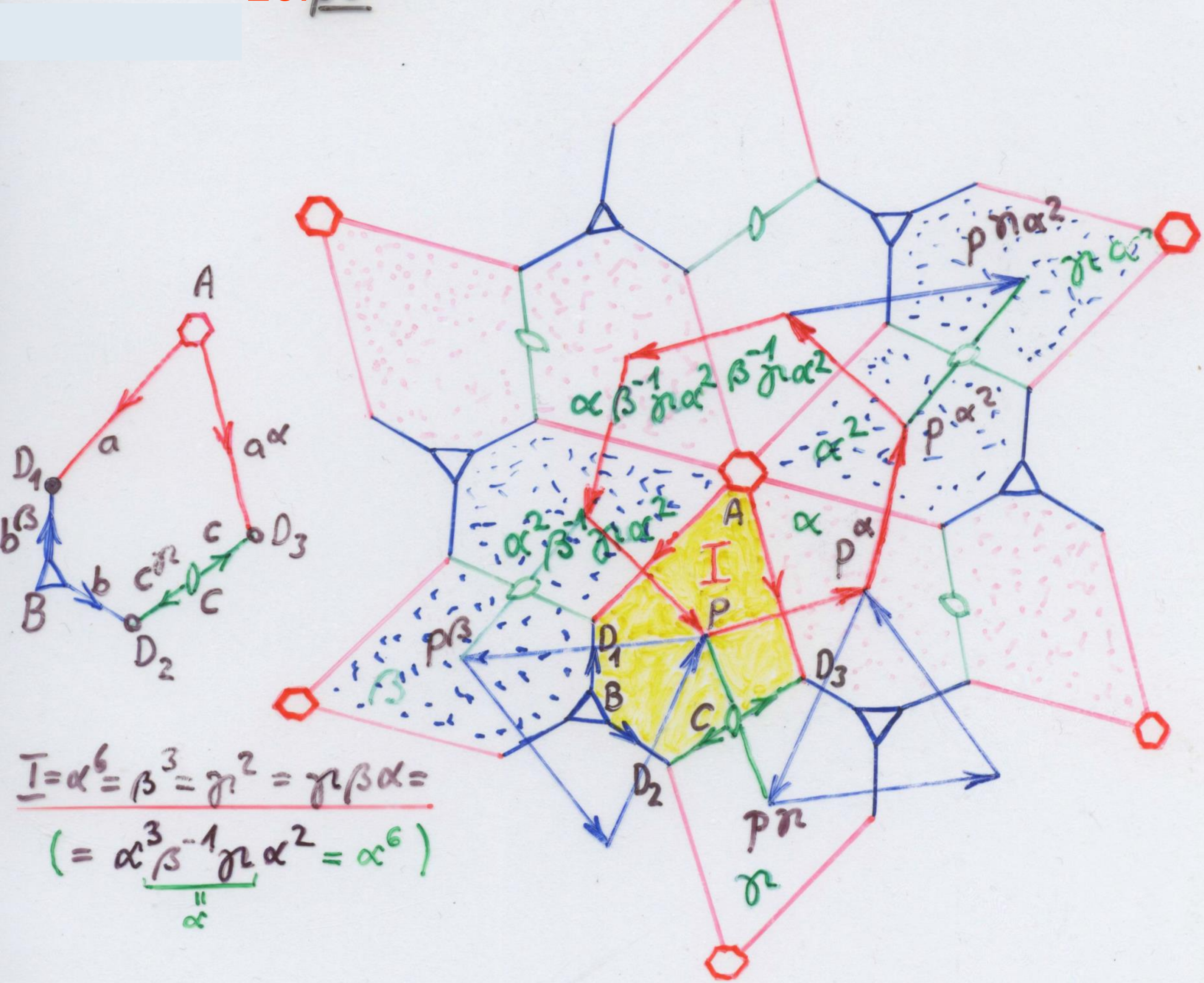
The same fund. domain for different trees of group

$\Gamma = (-2; [3]; \{(2)\}) = 3 * 2 \otimes^2$



16:p6

The Euclidean plane group **p6**, a fundamental domain and the group presentation



Thank you for your kind attention!

☺