Geometry of knots, links and polyhedra

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GEOMETRY AND SYMMETRY
Dedicated to Karoly Bezdek and Egon Schulte
on the occasion of their 60th birthdays
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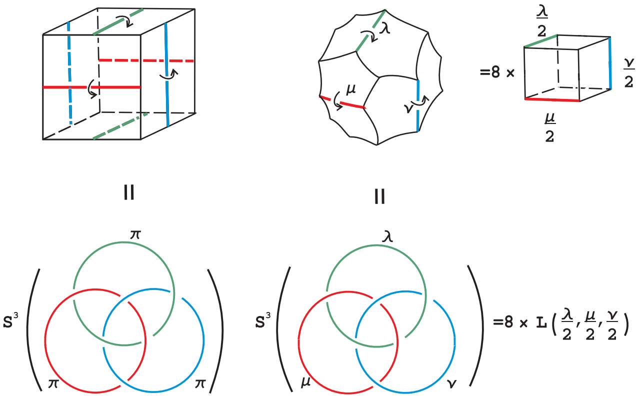
- Borromean Rings cone–manifold and Lambert cube

We start with a simple geometrical construction done by W. Thurston, D. Sullivan and J. M. Montesinos.



From polyhedra to knots and links

From the above consideration we get

$$Vol\, B(\lambda, \mu, \nu) = 8\, Vol\, L(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2}).$$

Recall that $B(\lambda, \mu, \nu)$ is

i) hyperbolic if $0 < \lambda, \mu, \nu < \pi$ **(E. M. Andreev)**

ii) Euclidean if $\lambda = \mu = \nu = \pi$

iii) spherical if $\pi < \lambda, \mu, \nu < 3\pi, \quad \lambda, \mu, \nu \neq 2\pi$
(R. Diaz, D. Derevnin and M.)

- Volume calculation for $L(\alpha,\beta,\gamma)$. The main idea.

0. Existence

$$L(\alpha,\beta,\gamma): \begin{cases} 0<\alpha,\beta,\gamma<\pi/2, & H^3 \\ \alpha=\beta=\gamma=\pi/2, & E^3 \\ \pi/2<\alpha,\beta,\gamma<\pi, & S^3. \end{cases}$$

1. Schläfli formula for $V = Vol\, L(\alpha,\beta,\gamma)$

$$k\mathrm{d}V = \frac{1}{2}(\ell_\alpha d\alpha + \ell_\beta d\beta + \ell_\gamma d\gamma), \quad k = \pm 1, 0$$

In particular in hyperbolic case:

$$\begin{cases} \frac{\partial V}{\partial \alpha} = -\frac{\ell_\alpha}{2}, \frac{\partial V}{\partial \beta} = -\frac{\ell_\beta}{2}, \frac{\partial V}{\partial \gamma} = -\frac{\ell_\gamma}{2} & (*) \\ V(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = 0. & (**) \end{cases}$$

2. Trigonometrical and algebraic identities

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh\ell_\alpha}=\frac{\tan\beta}{\tanh\ell_\beta}=\frac{\tan\gamma}{\tanh\ell_\gamma}=T\quad(\textbf{R.Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin\alpha}{\sinh\ell_\alpha}\frac{\sin\beta}{\sinh\ell_\beta}\frac{\cos\gamma}{\cosh\ell_\gamma}=1\quad(\textbf{Derevnin}-\textbf{Mednykh})$$

(iii)

$$\frac{T^2-A^2}{1+A^2}\,\frac{T^2-B^2}{1+B^2}\,\frac{T^2-C^2}{1+C^2}\,\frac{1}{T^2}=1,\quad(\textbf{HLM},\textbf{Topology}'\textbf{90})$$

where

$A=\tan\alpha, B=\tan\beta, C=\tan\gamma$. Equivalently,

$(T^2+1)(T^4-(A^2+B^2+C^2+1)T^2+A^2B^2C^2)=0.$

Remark. (ii) ⇒(i) and (i) & (ii) ⇒ (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha, \beta, \gamma)$ is given by

$$W = \frac{1}{4} \int_{T}^{\infty} \log \left(\frac{t^2 - A^2}{1 + A^2} \, \frac{t^2 - B^2}{1 + B^2} \, \frac{t^2 - C^2}{1 + C^2} \, \frac{1}{t^2} \right) \frac{\mathrm{d}t}{1 + t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A} \frac{\partial A}{\partial \alpha} = -\frac{1}{2} \arctan \frac{A}{T} = -\frac{\ell_\alpha}{2}.$$

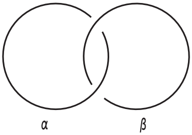
In a similar way

$$\frac{\partial W}{\partial \beta} = -\frac{\ell_\beta}{2} \quad \text{and} \quad \frac{\partial \mathrm{W}}{\partial \gamma} = -\frac{\ell_\gamma}{2}.$$

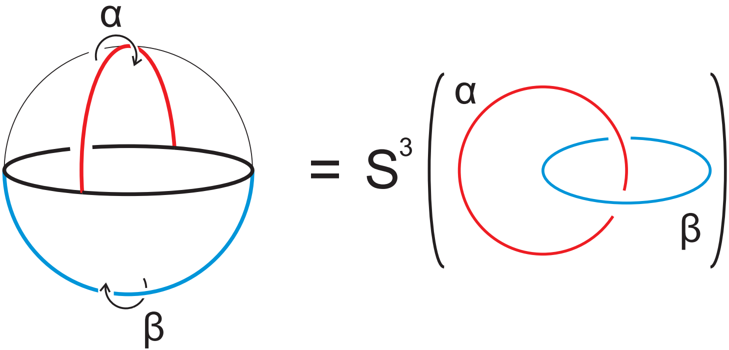
By convergence of the integral $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$. Hence, $W = V = Vol\, L(\alpha, \beta, \gamma)$.

- The Hopf link

The Hopf link 2_1^2 is the simplest two component link.



It has a few remarkable properties. First of all, the fundamental group $\pi_1(\mathbb{S}^3 \backslash 2_1^2) = \mathbb{Z}^2$ is a free Abelian group of rank two. It makes us sure that any finite covering of $\mathbb{S}^3 \backslash 2_1^2$ is homeomorphic to $\mathbb{S}^3 \backslash 2_1^2$ again. The second property is that the orbifold $2_1^2(\pi, \pi)$ arises as a factor space by $\mathbb{Z}_2$-action on the three dimensional projective space $\mathbb{P}^3$. That is, $\mathbb{P}^3$ is a two-fold covering of the sphere $\mathbb{S}^3$ branched over the Hopf link. In turn, the sphere $\mathbb{S}^3$ is a two-fold unbranched covering of the projective space $\mathbb{P}^3$.



Fundamental polyhedron $\mathcal{F}(\alpha, \beta)$ **for the cone-manifold** $2_1^2(\alpha, \beta)$.

Theorem 1

The Hopf link cone-manifold $2^2_1(\alpha,\beta)$ *is spherical for all positive* α *and* β. *The spherical volume is given by the formula* $\mathrm{Vol}\,(2^2_1(\alpha,\beta)) = \frac{\alpha\,\beta}{2}$.

Proof. Let $0 < \alpha, \beta \leqslant \pi$. Consider a spherical tetrahedron $\mathcal{T}(\alpha,\beta)$ with dihedral angles α and β prescribed to the top and bottom edges and with right angles prescribed to the remained ones. To obtain a cone-manifold $2^2_1(\alpha,\beta)$ we identify the faces of tetrahedron by α- and β-rotations in the respective edges. Hence, $2^2_1(\alpha,\beta)$ is spherical and

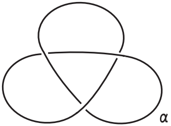
$$\mathrm{Vol}\,(2^2_1(\alpha,\beta)) = \mathrm{Vol}\,\mathcal{T}(\alpha,\,\beta) = \frac{\alpha\,\beta}{2}.$$

We note that $\mathcal{T}(\alpha,\,\beta)$ is a union of n^2 tetrahedra $\mathcal{T}(\frac{\alpha}{n},\,\frac{\beta}{n})$. Hence, for large positive α and β we also obtain

$$\mathrm{Vol}\,(2^2_1(\alpha,\beta)) = n^2\cdot\mathrm{Vol}\,\mathcal{T}(\frac{\alpha}{n},\,\frac{\beta}{n}) = \frac{\alpha\,\beta}{2}.$$

- The Trefoil

Let $\mathcal{T}(\alpha) = 3_1(\alpha)$ be a cone manifold whose underlying space is the three-dimensional sphere $\mathcal{S}^3$ and singular set is Trefoil knot $\mathcal{T}$ with cone angle α.



Since $\mathcal{T}$ is a toric knot by the Thurston theorem its complement $\mathcal{T}(0) = \mathcal{S}^3 \setminus \mathcal{T}$ in the $\mathcal{S}^3$ does not admit hyperbolic structure. We think this is the reason why the simplest nontrivial knot came out of attention of geometricians. However, it is well known that Trefoil knot admits geometric structure. H. Seifert and C. Weber (1935) have shown that the spherical space of dodecahedron (= Poincaré homology 3-sphere) is a cyclic 5-fold covering of $\mathcal{S}^3$ branched over $\mathcal{T}$.

Geometry of two bridge knots and links. The Trefoil.

Topological structure and fundamental groups of cyclic n-fold coverings have described by D. Rolfsen (1976) and A.J. Sieradsky (1986). In the case $\mathcal{T}(2\pi/n)$ $n \in \mathbb{N}$ is a geometric orbifold, that is can be represented in the form $\mathbb{X}^3/\Gamma$, where $\mathbb{X}^3$ is one of the eight three-dimensional homogeneous geometries and Γ is a discrete group of isometries of $\mathbb{X}^3$. By Dunbar (1988) classification of non-hyperbolic orbifolds has a spherical structure for $n \leq 5$, *Nil* for $n = 6$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ for $n \geq 7$. Quite surprising situation appears in the case of the Trefoil knot complement $\mathcal{T}(0)$. By P. Norbury (see Appendix A in the lecture notes by W. P. Neumann (1999)) the manifold $\mathcal{T}(0)$ admits two geometrical structures $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Theorem 2 (D. Derevnin, A. Mednykh and M. Mulazzani, 2008)

The Trefoil cone-manifold $\mathcal{T}(\alpha)$ *is spherical for* $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$. *The spherical volume of* $\mathcal{T}(\alpha)$ *is given by the formula*

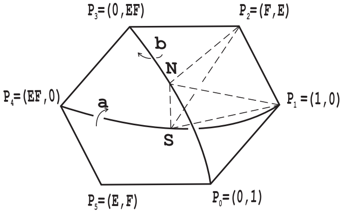
$$\mathrm{Vol}\,(\mathcal{T}(\alpha)) = \frac{(3\alpha - \pi)^2}{12}.$$

Proof. Consider $\mathbb{S}^3$ as the unite sphere in the complex space $\mathbb{C}^2$ endowed by the Riemannian metric

$$\mathrm{d}s_\lambda^2 = |\mathrm{dz}_1|^2 + |\mathrm{dz}_2|^2 + \lambda(\mathrm{dz}_1\mathrm{d}\bar{\mathrm{z}}_2 + \mathrm{d}\bar{\mathrm{z}}_1\mathrm{dz}_2),$$

where $\lambda = (2\sin\frac{\alpha}{2})^{-1}$. Then $\mathbb{S}^3 = (\mathbb{S}^3,\, \mathrm{d}s_\lambda^2)$ is the spherical space of constant curvature $+1$.

Then the fundamental set for $\mathcal{T}(\alpha)$ is given by the following polyhedron



where $E = e^{i\,\alpha}$ and $F = e^{i\,\frac{\alpha-\pi}{2}}$. The length ℓ_α of singular geodesic of $\mathcal{T}(\alpha)$ is given by $\ell_\alpha = |P_0P_3| + |P_1P_4| = 3\alpha - \pi$. By the Schläfli formula

$$\mathrm{dVol}\,\mathcal{T}(\alpha) = \frac{\ell_\alpha}{2}\mathrm{d}\alpha = \frac{3\alpha - \pi}{2}\mathrm{d}\alpha.$$

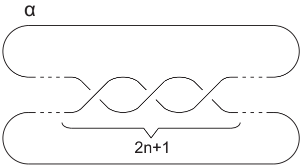
Hence,

$$\mathrm{Vol}\,\mathcal{T}(\alpha) = \frac{(3\alpha - \pi)^2}{12}.$$

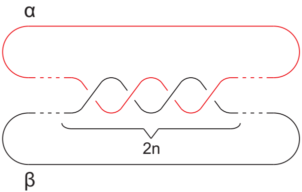
- Spherical structure on toric knots and links

The methods developed to prove Theorem 1 and Theorem 2 allowed to establish similar results for infinite families of toric knots and links. Consider the following cone–manifolds.

$T_n(\alpha)$



$T_n(\alpha,\beta)$



Geometry of two bridge knots and links

Theorem 3 (A. Kolpakov and M., 2009)

The cone-manifold $\mathcal{T}_n(\alpha), n \geq 1,$ *admits a spherical structure for*

$$\frac{2n-1}{2n+1}\pi < \alpha < 2\pi - \frac{2n-1}{2n+1}\pi$$

The length of the singular geodesics of $\mathcal{T}_n(\alpha)$ *is given by*

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

The volume of $\mathcal{T}_n(\alpha)$ *is equal to*

$$\mathrm{Vol}\,\mathcal{T}_n(\alpha) = \frac{1}{2n+1}\left(\frac{2n+1}{2}\alpha - \frac{2n-1}{2}\pi\right)^2.$$

Remark. The domain of the existence of a spherical metric in Theorem 3 was indicated earlier by J. Porti (2004).

Geometry of two bridge knots and links

Theorem 4 (A. Kolpakov and M., 2009)

The cone-manifold $\mathcal{T}_n(\alpha,\,\beta)$, $n \geq 2$, admits a spherical structure if the conditions

$$|\alpha - \beta| < 2\pi - \frac{2\pi}{n}, \quad |\alpha + \beta - 2\pi| < \frac{2\pi}{n}$$

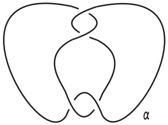
are satisfied. The lengths of the singular geodesics of $\mathcal{T}_n(\alpha,\,\beta)$ are equal to each other and are given by the formula

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - (n-1)\pi.$$

The volume of $\mathcal{T}_n(\alpha)$ is equal to

$$\mathrm{Vol}\,\mathcal{T}_n(\alpha\,\beta) = \frac{1}{2n}\left(\frac{\alpha + \beta}{2} n - (n-1)\pi\right)^2.$$

- The figure eight knot 4_1



It was shown in Thurston lectures notes that the figure eight compliment $\mathbb{S}^3 \setminus 4_1$ can be obtained by gluing two copies of a regular ideal tetrahedron. Thus, $\mathbb{S}^3 \setminus 4_1$ admits a complete hyperbolic structure. Later, it was discovered by A. C. Kim, H. Helling and J. Mennicke that the $n-$ fold cyclic coverings of the 3-sphere branched over 4_1 produce beautiful examples of the hyperbolic Fibonacci manifolds. Theirs numerous properties were investigated by many authors. 3-dimensional manifold obtained by Dehn surgery on the figure eight compliment were described by W. P. Thurston. The geometrical structures on these manifolds were investigated in Ph.D. thesis by C. Hodgson.

The following result takes a place due to Thurston, Kojima, Hilden, Lozano, Montesinos, Rasskazov and M..

Theorem 5

A cone-manifold $4_1(\alpha)$ *is hyperbolic for* $0 \leq \alpha < \alpha_0 = \frac{2\pi}{3}$, *Euclidean for* $\alpha = \alpha_0$ *and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$.

Other geometries on the figure eight cone-manifold were studied by C. Hodgson, W. Dunbar, E. Molnar, J. Szirmai and A. Vesnin.

The volume of the figure eight cone-manifold in the spaces of constant curvature is given by the following theorem.

Theorem 6 (A. Rasskazov and M., 2006)

Let $\mathrm{V}(\alpha) = \mathrm{Vol}\ 4_1(\alpha)$ *and* ℓ_α *is the length of singular geodesic of* $4_1(\alpha)$. *Then*

$(\mathbb{H}^3)$ $\mathrm{V}(\alpha) = \int_\alpha^{\alpha_0} \mathrm{arccosh}\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ 0 \le \alpha < \alpha_0 = \frac{2\pi}{3}$,

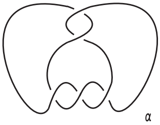
$(\mathbb{E}^3)$ $\mathrm{V}(\alpha_0) = \frac{\sqrt{3}}{108}\,\ell_{\alpha_0}^3$,

$(\mathbb{S}^3)$ $\mathrm{V}(\alpha) = \int_{\alpha_0}^{\alpha} \arccos\,(1 + \cos\theta - \cos 2\theta)\mathrm{d}\theta,\ \alpha_0 < \alpha \le \pi,\quad \mathrm{V}(\pi) = \frac{\pi^2}{5}$,

$\mathrm{V}(\alpha) = 2\mathrm{V}(\pi) - \mathrm{V}(2\pi - \alpha),\ \pi \le \alpha < 2\pi - \alpha_0.$

- The 5_2 knot

The knot 5_2 is a rational knot of a slope $7/2$.



Historically, it was the first knot which was related with hyperbolic geometry. Indeed, it has appeared as a singular set of the hyperbolic orbifold constructed by L.A. Best (1971) from a few copies of Lannér tetrahedra with Coxeter scheme $\circ \equiv \circ - \circ = \circ$. The fundamental set of this orbifold is a regular hyperbolic cube with dihedral angle $2\pi/5$. Later, R. Riley (1979) discovered the existence of a complete hyperbolic structure on the complement of 5_2. In his time, it was one of the nine known examples of knots with hyperbolic complement.

Geometry of two bridge knots and links. 5_2– knot.

A few years later, it has been proved by W. Thurston that all non-satellite, non-toric prime knots possess this property. Just recently it became known (2007) that the Weeks-Fomenko-Matveev manifold $\mathcal{M}_1$ of volume 0.9427... is the smallest among all closed orientable hyperbolic three manifolds. We note that $\mathcal{M}_1$ was independently found by J. Przytycki and his collaborators (1986). It was proved by A. Vesnin and M. (1998) that manifold $\mathcal{M}_1$ is a cyclic three fold covering of the sphere $\mathbb{S}^3$ branched over the knot 5_2. It was shown by J. Weeks computer program Snappea and proved by Moto-O Takahahsi (1989) that the complement $\mathbb{S}^3 \setminus 5_2$ is a union of three congruent ideal hyperbolic tetrahedra.

The next theorem has been proved by A. Rasskazov and M. (2002), R. Shmatkov (2003) and J. Porti (2004) for hyperbolic, Euclidian and spherical cases, respectively.

Theorem 7

A cone manifold $5_2(\alpha)$ *is hyperbolic for* $0 \leq \alpha < \alpha_0$, *Euclidean for* $\alpha = \alpha_0$, *and spherical for* $\alpha_0 < \alpha < 2\pi - \alpha_0$, *where* $\alpha_0 \simeq 2.40717$ *is a root of the equation*

$$-11 - 24\cos(\alpha) + 22\cos(2\alpha) - 12\cos(3\alpha) + 2\cos(4\alpha) = 0.$$

Theorem 8 (A. Mednykh, 2009)

Let $5_2(\alpha),\ 0 \le \alpha < \alpha_0$ *be a hyperbolic cone-manifold. Then the volume of* $5_2(\alpha)$ *is given by the formula*

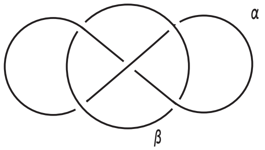
$$\mathrm{Vol}\,(5_2(\alpha)) = i \int_{\bar{z}}^{z} \log\left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2}\right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot\frac{\alpha}{2}$ *and* z, $\Im z > 0$ *is a root of equation*

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

A new and completely different approach to find volume of the above cone-manifold is contained in our resent paper (Ji-Young Ham, Alexander Mednykh, Vladimir Petrov, 2014).

- The Whitehead link 5_1^2



The ten smallest closed hyperbolic $3-$ manifolds can be obtained as the result of Dehn surgery on components of the Whitehead link (**P. Milley, 2009**). All of them are two-fold coverings of the $3-$ sphere branched over some knots and links (**A. Vesnin and M., 1998**).

The Whitehead link.

Theorem 9 (A. Vesnin and M., 2002)

Let $W(\alpha, \beta)$ be a hyperbolic Whitehead link cone-manifold. Then the volume of $W(\alpha, \beta)$ is given by the formula

$$i \int_{\bar{z}}^{z} \log\left(\frac{2(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta^2 - \zeta^3)}\right) \frac{d\zeta}{\zeta^2 - 1},$$

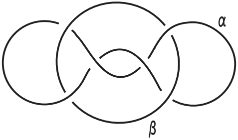
where $A = \cot\frac{\alpha}{2}$, $B = \cot\frac{\beta}{2}$ *and* z, $\Im(z) > 0$ *is a root of the equation*

$$2(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z^2 - z^3).$$

A similar result as valid also in spherical geometry. The Euclidean volume of $W(\alpha, \beta)$ was calculated by **R. Shmatkov, 2003**.

Geometry of the twist links.

- The Twist link 6_3^2



Theorem 10 (D. Derevnin, M Mulazzani and M., 2004)

Let $6_3^2(\alpha, \beta)$ be a hyperbolic cone-manifold. Then the volume of $6_3^2(\alpha, \beta)$ is given by the formula

$$i \int_{\overline{z}}^{z} \log \left[\frac{4(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta - \zeta^2)^2} \right] \frac{d\zeta}{\zeta^2 - 1}.$$

where $A = \cot \frac{\alpha}{2}$, $B = \cot \frac{\beta}{2}$, and z, $\Im(z) > 0$ is a root of the equation

$$4(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z - z^2)^2.$$

The volumes of more complicated twist links are obtained in our recent joint work with Koya Shimokawa, Saitama University and Yokota Yoshiyuki, Tokyo Metropolitan University (2015). Consider the Stevedore knot 6_1. Then we have

Theorem

The volume of the hyperbolic cone-manifold $6_1(\alpha)$ is given by integral

$$i\int_{\overline{z}}^{z}\log\left[\frac{8(\zeta^2+A^2)}{(1+A^2)(1-\zeta)(2+\zeta+\zeta^2-(1-\zeta)\sqrt{2+2\zeta+\zeta^2})}\right]\frac{d\zeta}{\zeta^2-1},$$

where $A=\cot\frac{\alpha}{2}$ and z and $\overline{z}$ are complex conjugated roots of the integrand.