Results and Problems from Minkowski Geometry

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Joint work with V. Boltyanski, M. Lassak, M. Spirova, K. J. Swanepoel, and Senlin Wu

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Minkowski Geometry

Philosophy is a game with aims, but without rules. Mathematics is a game with rules, but without aims. Ian Ellis

Minkowski Geometry is the *geometry* of finite dimensional real Banach spaces (= Minkowski spaces)

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Historical origins:

- B. Riemann (1868): ℓ_4-norm
- H. Minkowski (1896): axiomatic approach
- D. Hilbert (~ 1900): 4th Hilbert problem
- St. Banach, H. Busemann, V. Klee, Br. Grünbaum, ...

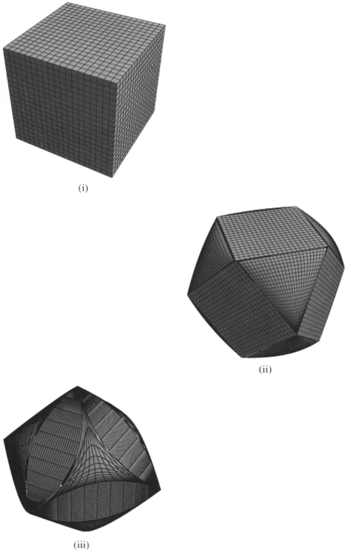
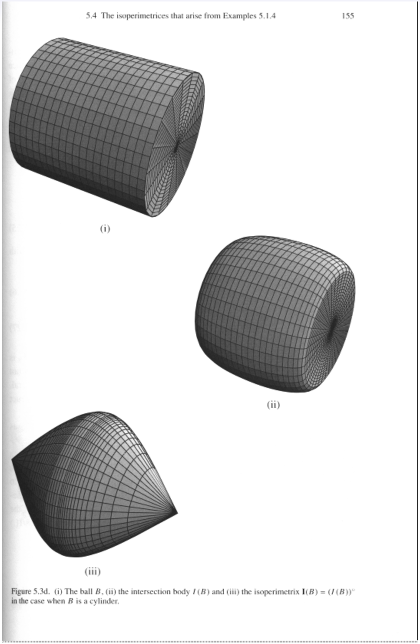


Figure 5.3b. (i) The ball B, (ii) the intersection body $I(B)$ and (iii) the isoperimetrix $\mathbf{I}(B) = (I(B))^{\circ}$ in the case when B is a cube.



5.4 The isoperimetrices that arise from Examples 5.1.4 155

(i)

(ii)

(iii)

Figure 5.3d. (i) The ball B, (ii) the intersection body $I(B)$ and (iii) the isoperimetrix $\mathbf{I}(B) = (I(B))^{\circ}$ in the case when B is a cylinder.

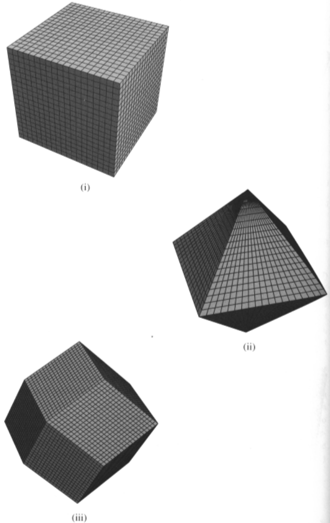
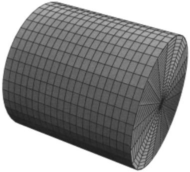
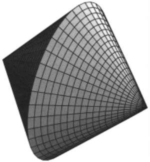


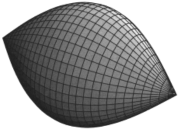
Figure 5.4b. (i) The ball B, (ii) the dual ball B° and (iii) the isoperimetrix $\mathbf{I}(B) = \Pi(B^{\circ})$ in the case when B is a cube.



(i)



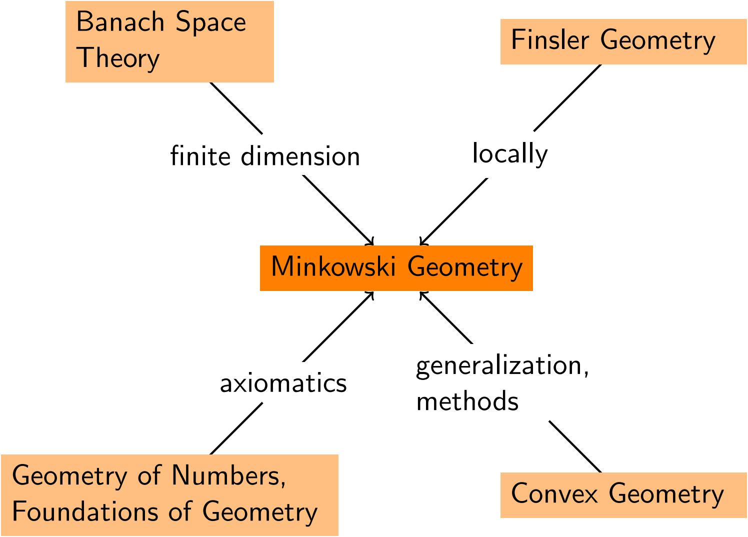
(ii)



(iii)

Figure 5.4d. (i) The ball B, (ii) the dual ball B° and (iii) the isoperimetrix $\mathbf{I}(B) = \Pi(B^\circ)$ in the case when B is a cylinder. Note that in this case $\mathbf{I}(B)$ is the solid of revolution of the cosine curve; see Chilton and Coxeter [110] and [262].

Connections



Today

Revitalization within "other" research fields, such as

- Computational Geometry
- Discrete Geometry
- Functional Analysis
- Optimization (Location Science, . . .)
- Convex Analysis
- non-Euclidean geometries in general

$\vdots$

Motivation

- direction-dependent phenomena (e.g., in Physics), such as crystal growth, whiskers, etc.
- Geometry of Numbers (lattice point problems)
- Location Science with non-Euclidean road systems

The topic of Pure Physics is the development of the rules of the understandable world. The topic of Pure Mathematics is the development of the rules of the human intelligence.
J.J. Sylvester

Concept

$\mathbb{M}^d = (\mathbb{R}^d, \|\cdot\|)$ . . . d-dimensional *Minkowski space*
unit ball B of $\mathbb{M}^d$ . . . convex body centered at the origin
$\|\cdot\|$ . . . *norm* induced by B:

$$\forall x \in \mathbb{R}^d \colon \|x\| = \min\{\lambda \geq 0 \colon x \in \lambda B\}$$

1. Minkowski Geometry and Convexity

- H. Minkowski (1896): Geometry of numbers
- St. Banach (early 20's): Foundations of Functional Analysis
- H. Busemann (since 40's): Finsler Geometry
- M.M. Day, R.C. James, D. Amir et al. (since 40's): finite dimensional Banach Space Theory

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- H.G. Eggleston, C.M. Petty, Br. Grünbaum, V. Klee, G.D. Chakerian, K. Leichtweiss, . . . (since 50's): Convex Geometry
- H. Groemer, R. Schneider, K. Ball, A.C. Thompson, . . . (since 80's): Convex Geometry

1.1. Partial fields

- special norms (ellipsoids, ℓ_p-norms, zonoids, polytopes, ...)
- special types of convex bodies studied in normed spaces (complete sets, bodies of constant width, reduced bodies, ball polytopes, ...)
- projection bodies and intersection bodies (isoperimetric problem)
- concepts of area and content (e.g., Holmes-Thompson area)

1.2. Reduced bodies in $\mathbb{M}^d$

Poetry is the art of giving different names to the same thing;
Math is the art of giving the same name to different things.
Henri Poincaré

K	. . . compact set in $\mathbb{M}^d$
$\mathrm{diam}(K)$	. . . maximal width = *diameter* of K
$\Delta(K)$	. . . minimal width = *thickness* of K

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$\Delta(K)$	...minimal width = *thickness* of K

Definition 1.1

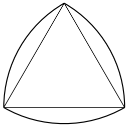
A convex body $C \subset \mathbb{M}^d$ is said to be *complete* if $\mathrm{diam}(K) > \mathrm{diam}(C)$ for any compact set K with $C \subsetneq K$.

Definition 1.2

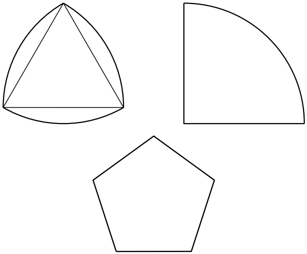
A convex body $R \subset \mathbb{M}^d$ is said to be *reduced* if $\Delta(K) < \Delta(R)$ for any *convex* body K with $K \subsetneq R$.

Euclidean space $\mathbb{R}^d$

complete $=$ constant width

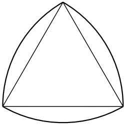


reduced

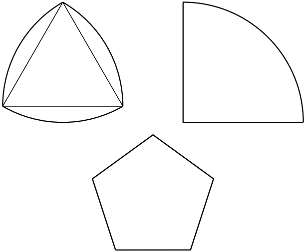


Euclidean space $\mathbb{R}^d$

complete = constant width



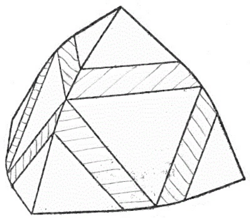
reduced



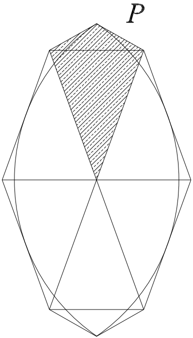
- Are there reduced polytopes in $\mathbb{R}^d$, $d \geq 3$?
- Is a strictly convex reduced body in $\mathbb{R}^d$, $d \geq 3$, necessarily of constant width?

Euclidean space $\mathbb{R}^d$

- Which convex body $R \subset \mathbb{R}^d$, $d \geq 3$, of thickness $\Delta(R) = 1$ has minimal volume?



Minkowski space $\mathbb{M}^d$



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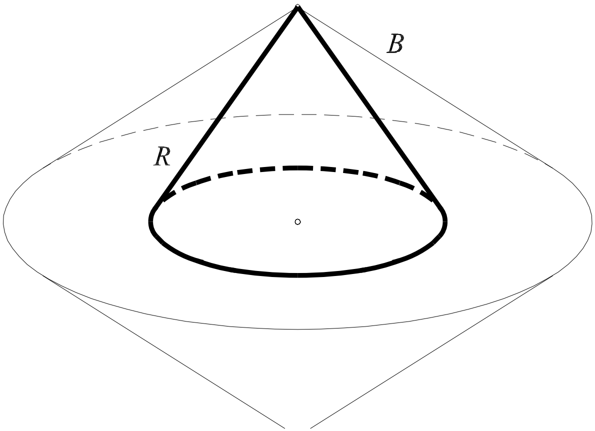


Some results

Let R be a reduced body in $\mathbb{M}^d$, $d \geq 2$.

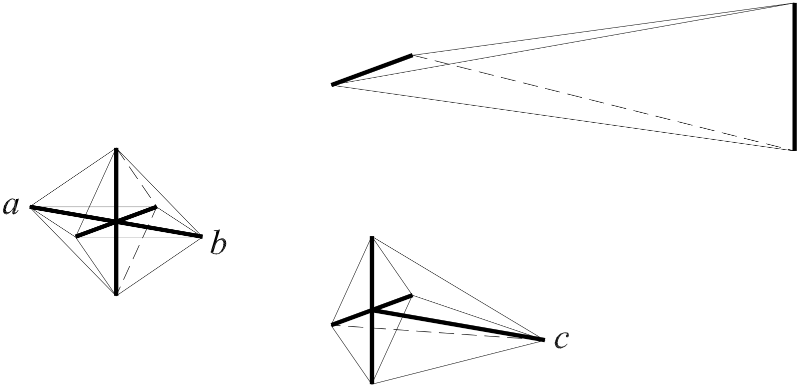
$$R \text{ centrally symmetric} \quad \Leftrightarrow \quad R \text{ and } B \text{ are homothets}$$

$$R \text{ smooth} \quad \Leftrightarrow \quad R \text{ of constant width}$$

$$\text{No } R \text{ is smooth} \quad \Leftarrow \quad B \text{ is not smooth}$$

Some results

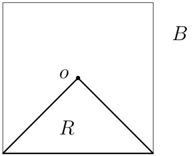
Any R is representable as homothet of $\operatorname{conv}\{t_1 + [o, b_1], \ldots, t_m + [o, b_m]\}$ $\Leftarrow$ $B = \operatorname{conv}\{\pm b_1, \ldots, \pm b_m\}$ is a polytope



Results in the Minkowski plane

for $d = 2$:

- For every reduced body $R \subset \mathbb{M}^2$ we have $\frac{\mathrm{diam}(R)}{\Delta(R)} \leq 2$.

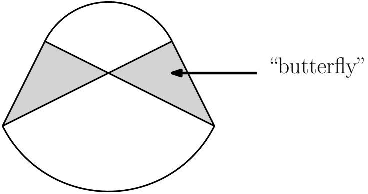


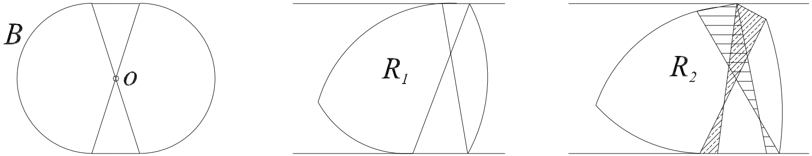
- Every strictly convex reduced body $R \subset \mathbb{M}^2$ is of constant width.

The basic principle of modern Mathematics is to achieve a complete fusion of "geometric" and "analytic" ideas.
J.A.E. Dieudonné

Results in the Minkowski plane

- The boundary of every reduced body $R \subset \mathbb{M}^2$ is the union of all arms of butterflies of R and all endpoints of thickness chords of R.



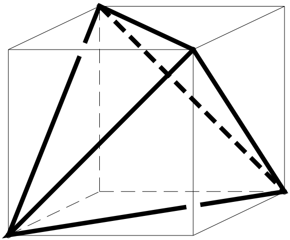


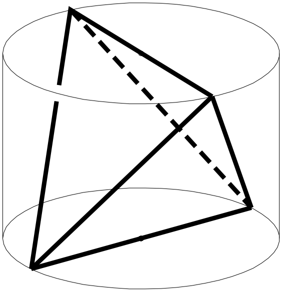
Results in the Minkowski plane

- A triangle in a normed plane is reduced iff it is equilateral in the respective antinorm (whose unit ball is the dual of B rotated by $90°$). ($\longrightarrow$ Fermat-Torricelli problem, Steiner minimum trees)

- The perimeter of every planar reduced body R satisfies $\mathrm{per}(R) \geq \pi \cdot \Delta(R)$, with equality iff R is of constant width. ($\mathrm{per}(R) > \pi \cdot \Delta(R)$ if R is a polygon.)

Minkowski space $\mathbb{M}^d$





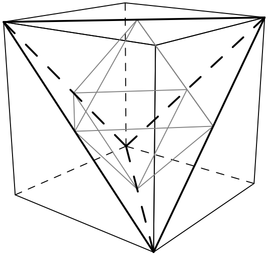
Minkowski space $\mathbb{M}^d$

Problems:

- Do there exist normed spaces in which the balls are the only reduced bodies?
- In which normed spaces is any reduced body also complete, or even of constant width?
- Do there exist normed spaces $\mathbb{M}^d$, $d \geq 3$, in which there are no reduced *polytopes*?

Minkowski space $\mathbb{M}^d$

- Which convex bodies of thickness 1 have minimal volume (for any $\mathbb{M}^d$)?
- Since completeness is not equivalent to constant width in Minkowski spaces, there are also complete sets which are not reduced for $d \geq 3$.



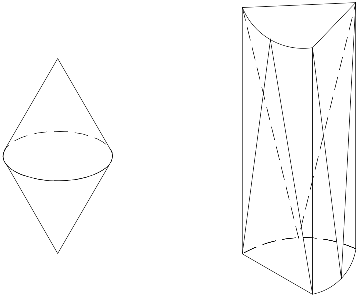
Minkowski space $\mathbb{M}^d$

$\Rightarrow$ Give a complete Venn diagram representation of the families of reduced, complete and constant with bodies in $\mathbb{M}^d, d \geq 3$.

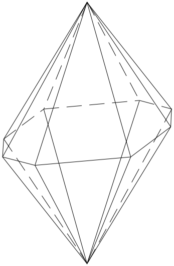
The art of doing Mathematics consists in finding that special case which contains all the germs of generality.
David Hilbert

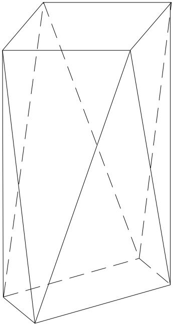
Conjecture

In any normed space $\mathbb{M}^d$, $d \geq 3$, there exist reduced bodies R with $\Delta(R) = 1$ having arbitrarily large diameter!



Everything that is only likely is most likely wrong!
René Descartes





2. Discrete & Computational Geometry in normed spaces

In the great garden of Geometry, everyone can pick up a bunch of flowers, simply following his taste. David Hilbert

- H. Minkowski (1896): Geometry of numbers
- H. Hadwiger, V. Boltyanski, L. Fejes Tóth, B. Grünbaum, V. Klee et al. (since 60's): packings & coverings, circum- and inballs, Borsuk numbers

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- R. Klein, F. Aurenhammer, F. Santos et al. (since 90's): bisectors and Voronoi diagrams, geometric dilation, ...
- V. Klee, P. Gritzmann (since 90's): Computational Convexity
- today: Chebyshev sets (also computationally), minmax and minsum location problems, ball operators, ...

2.1. The Fermat-Torricelli problem in $\mathbb{M}^d$

Women demand what is impossible: we should forget their age, but not their birthday! Karl Farkas

$\mathbb{E}^d$: P. de Fermat (1638), E. Torricelli (1640)

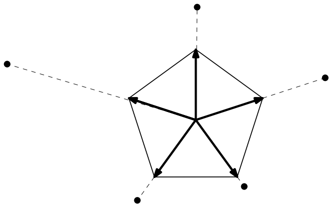
Given an arbitrary finite point set $\{x_1, \ldots, x_m\} \subset \mathbb{E}^d$, find the (unique) point $x_{min} \in \mathbb{E}^d$ for which

$$f(x) = \sum_{i=1}^{m} \|x_i - x\|, x \in \mathbb{E}^d,$$

is minimal!

- unique solution in the

floating case
$m = 5$



absorbed case
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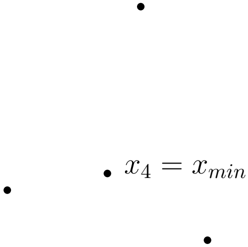

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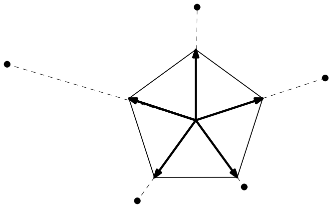

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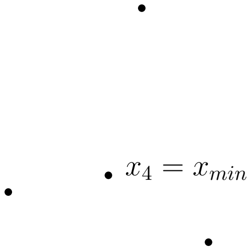
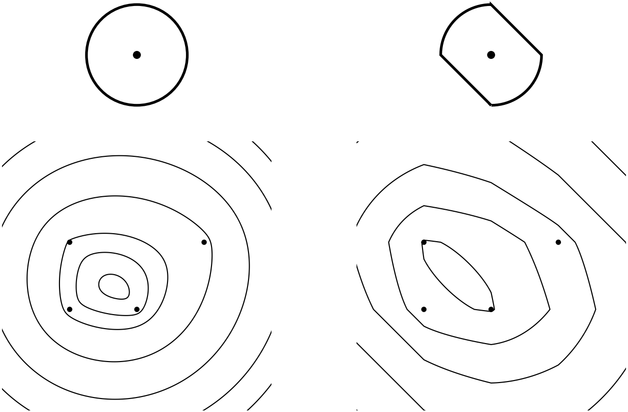


- $m \geq 5$: no exact ruler-and-compass construction (via Galois theory)

- level curves of $f(x)$: multifocal ellipses etc.



First important tool for $\mathbb{M}^d$

d-segment from x to y, $x, y \in \mathbb{M}^d$:

$$[x, y]_d := \{z \in \mathbb{M}^d \colon \|x - z\| + \|z - y\| = \|x - y\|\}$$

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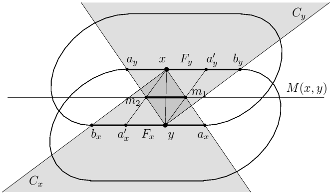

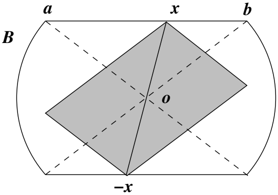

Second important tool for $\mathbb{M}^d$

The *norming functional* of $x \in \mathbb{M}^d$ is a $\phi \in (\mathbb{M}^d)^*$ such that the *dual norm* $\|\phi\| = \max_{\|x\|=1} \phi(x) = 1$ and $\phi(x) = \|x\|$. The hyperplane $\phi^{-1}(1) = \{y \in \mathbb{M}^d \colon \phi(y) = 1\}$ is then the supporting hyperplane of B at x.

$\Rightarrow \mathbb{M}^d$ is smooth $\Leftrightarrow$ Each $x \neq o$ has a unique norming functional.

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$\Rightarrow \mathbb{M}^d$ is smooth $\Leftrightarrow$ Each $x \neq o$ has a unique norming functional.

Definition 2.1

The set of all $x_0 \in \mathbb{M}^d$ minimizing $\sum_{i=1}^{m} \|x - x_i\|, x \in \mathbb{M}^d$, is called the *Fermat-Torricelli locus* $FT\{x_1, \ldots, x_m\}$ of $\{x_1, \ldots, x_m\}$.

Theorem 2.1

Let $X := \{x_0, x_1, \ldots, x_m\} \subset \mathbb{M}^d$ *be given arbitrarily.*

1. *If* $x_0 \neq x_1, \ldots, x_m$, *then* $X \setminus \{x_0\}$ *is a floating FT configuration with respect to* x_0 *iff each* $x_i - x_0$ *has a norming functional* ϕ_i *such that*

$$\sum_{i=1}^{m} \phi_i = o.$$

2. *If* $x_0 = x_j$ *for some* $j \in \{1, \ldots, m\}$, *then* $X \setminus \{x_0\}$ *is an absorbing FT configuration with respect to* x_0 *iff each* $x_i - x_0$ $(i \neq j)$ *has a norming functional* ϕ_i *such that*

$$\left\| \sum_{i=1}^{m} \phi_i \right\| \leq 1 \text{ for } i \neq j.$$

Given a functional $\phi \in (\mathbb{M}^d)^*$ of norm 1 and some $x \in \mathbb{M}^d$. Define the *cone*

$$C(x, \phi) = x - \{a \colon \phi(a) = \|a\|\}.$$

Thus, $C(x, \phi)$ is the translate by x of the union of all rays from o through the exposed face of $\phi^{-1}(-1) \cap B$.

Theorem 2.2

For any $\mathbb{M}^d$, let $p \in FT\{x_1, \ldots, x_m\} \setminus \{x_1, \ldots, x_m\}$, and with ϕ_i as norming functional of $x_i - p$ let $\sum_{i=1}^m \phi_i = o$. Then

$$FT\{x_1, \ldots, x_m\} = \bigcap_{i=1}^{m} C(x_i, \phi_i)$$

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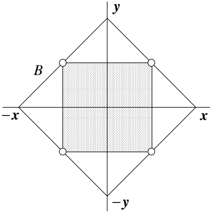
$$FT\{x_1, \ldots, x_m\} = \bigcap_{i=1}^{m} C(x_i, \phi_i)$$

Corollary 2.1

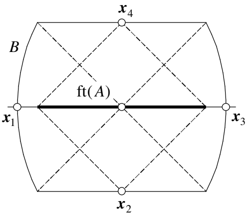
If $\{x_1, \ldots, x_{2k}\}$ can be matched up to form k d-segments $[x_i, x_{k+i}]_d, i = 1, \ldots, k$, with non-empty intersection, then

$$FT\{x_1, \ldots, x_{2k}\} = \bigcap_{i=1}^{k} [x_i, x_{k+i}]_d.$$

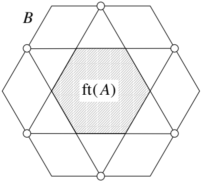
Examples



$FT(A) = \mathrm{conv}(A)$ is possible in the maximum norm.



Two non-disjoint d-segments create $FT(A)$.



Three non-disjoint d-segments create $FT(A)$.

Theorem 2.3

For any $A \subset \mathbb{M}^d$ *we have* $|A \cap FT(A) \leq 2^d|$. *Equality holds iff* $\mathbb{M}^d$ *is isometric to* ℓ_1^d, *with* $A \cap FT(A)$ *corresponding to a homothet of the Hamming cube* $\{0, 1\}^d$.

2.2. Universal covers in normed planes

If we knew what it was we were doing, it would not be called research, would it? Albert Einstein

- $p \in \mathbb{M}^2$ is *Birkhoff orthogonal* to $q \in \mathbb{M}^2$ if, for any $\lambda \in \mathbb{R}$, $\|p\| \leq \|p + \lambda q\|$ holds (not symmetric!)

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- Busemann (1947): For any $\|\cdot\|$, there is an *antinorm* $\|\cdot\|_a$ such that q is Birkhoff orthogonal to p with respect to $\|\cdot\|_a$
- the unit circle C_a of $\|\cdot\|_a$ is the *isoperimetrix* of $\|\cdot\|$
- if C and C_a are homothetic, then $\mathbb{M}^2$ is a *Radon plane*
- if follows that Birkhoff orthogonality is symmetric iff $\mathbb{M}^2$ is a Radon plane

Definition 2.2 (Lebesgue, 1914)

A set of smallest area containing a congruent (translative) copy of any planar set of diameter 1 is said to be a *universal cover* (*strong universal cover*).

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Lemma 2.1

Every planar set of diameter 1 *is contained in some planar set of constant width* 1*.*

Theorem 2.4

In any normed plane, an anti-regular hexagon circumscribed about a circle of diameter 1 *is a strong universal cover.*

Theorem 2.5

In any normed plane, an anti-regular triangle circumscribed about a circle of diameter 1 *is a strong universal cover.*

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In any normed plane, an anti-regular triangle circumscribed about a circle of diameter 1 *is a strong universal cover.*

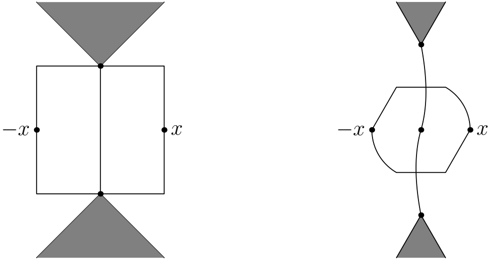
Theorem 2.6

In any normed plane, there is a square of side length 1 *which is a strong universal cover. If any such square is a strong universal cover, then the plane is a Radon plane.*

Problem: Find more geometric figures which are universal covers in any normed plane (or large norm classes), also in higher dimensions!

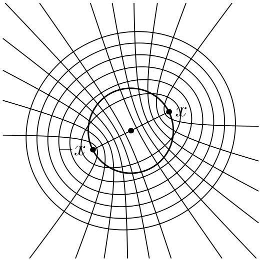
3. Some curve theory in normed planes

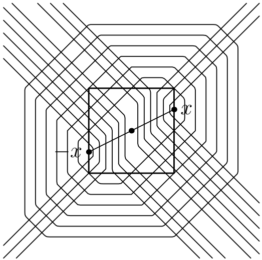
Everyone knows what a curve is, until he has studied enough Mathematics to become confused through the countless number of possible exceptions. Felix Klein



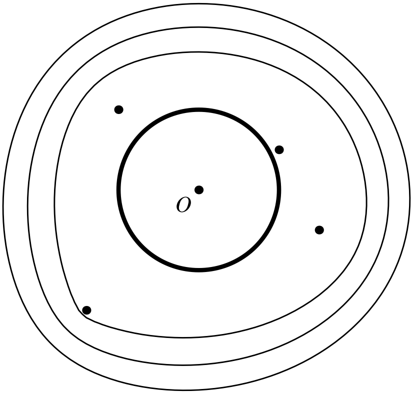
3. Some curve theory in normed planes

- C.M. Petty, H. Guggenheimer (50's): Frenet formulas, involutes & evolutes, 4-vertex-theorem
- S. Tabachnikov, M. Ghandehari (90's): Minkowskian caustics, curvature functions
- R. Ait-Haddou et al. (2000): applications in CAD (freeform curve design etc.)



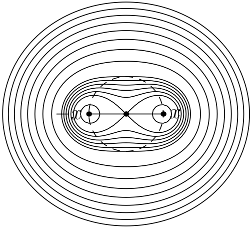


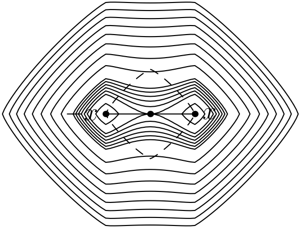
Euclidean multifocal ellipses



3.1. Cassini curves in normed planes

...locus of all points in $\mathbb{E}^2$ whose *product* of distances to two given points (*foci*) is constant





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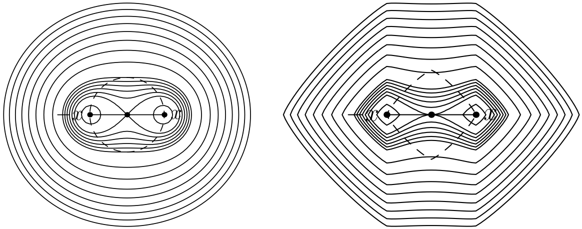

G.D. Cassini (1680): proposed these curves (in opposition to J. Kepler) as planetary orbits

Geometry is knowledge of the eternally existent. Pythagoras

Definitions

The point set

$$C(o, 2x, c) := \{z \in \mathbb{M}^2 \colon \|z\| \cdot \|z - 2x\| = c\}$$

is called the *Cassini curve with foci* $o, 2x$ and *of size* $c > 0$.

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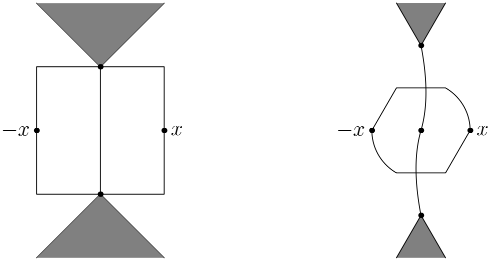
is called the *Cassini curve with foci* $o, 2x$ and *of size* $c > 0$. Furthermore, $B(-x, x) := \{y \in \mathbb{M}^2 \colon \|y - (-x)\| = \|y - x\|\}$ is called the *bisector* of $\mathrm{conv}\{-x, x\}$, $-x \neq x$, and we introduce

$$H^{\mp}_{-x,x} := \left\{ y \in \mathbb{M}^2 \colon \|y - (-x)\| \begin{matrix} \leq \\ \geq \end{matrix} \|y - x\| \right\}.$$



Theorem 3.1

For $x \in \mathbb{M}^2$ with $\|x\| = 1$ and $c > 0$ the set $C(o, 2x, c) \cap B(o, 2x)$ is symmetric with respect to x, and it is the union of two (possibly degenerate) segments.

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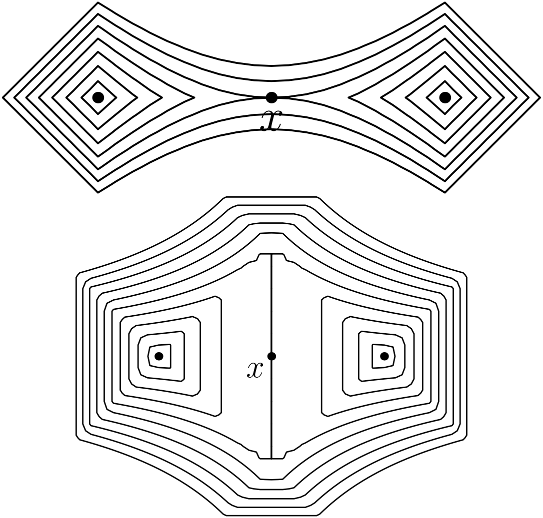
Theorem 3.2

For $x \in \mathbb{M}^2$ with $\|x\| = 1$ and $0 < c < 1$, $C(o, 2x, c)$ is the union of two disjoint closed curves which bound two disjoint star-shaped sets.


x
x

Theorem 3.3

For $x \in \mathbb{M}^2$ with $\|x\| = 1$ each of the two sets $C(o, 2x, 1) \cap H^{\mp}_{o,2x}$ is a closed curve, and we have

$$\emptyset \neq \left(C(o, 2x, 1) \cap H^{-}_{o,2x}\right) \cap \left(C(o, 2x, 1) \cap H^{+}_{o,2x}\right) \subset B(o, 2x).$$

In general, this set may contain infinitely many points. $\mathbb{M}^2$ is strictly convex if this set is a singleton for each unit vector.

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Theorem 3.4

For $x \in \mathbb{M}^2$ with $\|x\| = 1$ and $c > 1$, then $C(o, 2x, c)$ is a simple closed curve symmetric with respect to x.

Remark 3.1

One can construct normed planes such that the region bounded by $C(o, 2x, c)$ is not convex for any $c \geq 1$.

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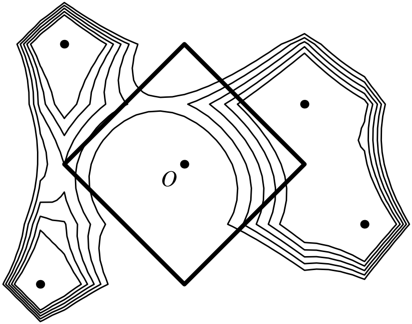
Observation: In the literature, *multifocal Cassini curves* were never studied for normed planes, although for the Euclidean norm they are very popular (e.g., as so-called *n*-lemniscates).



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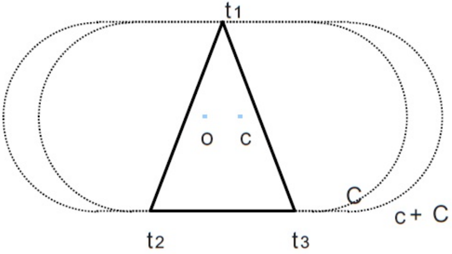
- not much is known about conic sections (different definitions can describe different classes of curves)
- the geometry of circles is not satisfactorily developed
- Radon curves and equiframed curves as tools
- (almost) *nothing* is known about further individual curve classes in normed planes !!

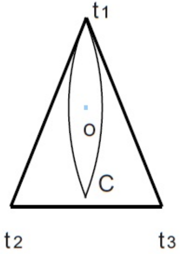
4. Elementary Geometry in Minkowski spaces

How to prove something we learn in Elementary Geometry!

- S. Gołab, D. Laugwitz (since 30's): "Minkowskian π"
- L. Tamassy, Br. Grünbaum (since 60's): extensions to gauges, Feuerbach circle, 3-circles-theorem, ...
- L.M. Kelly, I.M. Yaglom, J.J. Schäffer: re-entrant property, polygons inscribed in Jordan curves, circle geometry, ...
- today: *many* theorems from $\mathbb{E}^2$ were never generalized

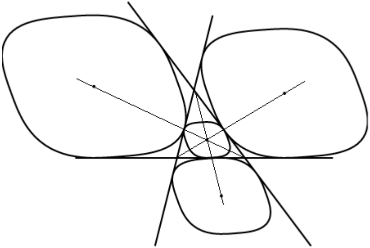
A triangle with a non-unique circumcircle (smooth norm) and a triangle without circumcircle (strictly convex norm)

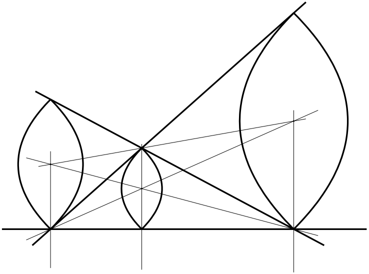




Without Geometry, life would be pointless!

A triangle with three and a triangle with only two excircles





Some results

It is the enjoyment of shape in a higher sense that makes the geometer. Alfred Clebsch

Theorem 4.1

Let p_1, p_2, p_3 be three distinct points on the unit circle C of a strictly convex normed plane, and let $x_i + C$ $(i = 1, 2, 3)$ be three circles different from C, each containing two points p_i. Then $\bigcap_{j=1}^{3}(x_j + C)$ *consists of precisely one point p.*

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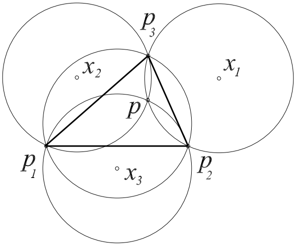
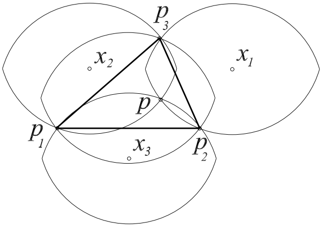
Definition 4.1

The point p is called the *C-orthocenter* of the triangle $p_1p_2p_3$ since $p - p_i$ is *James orthogonal* to $p_j - p_k$ $(\{1, 2, 3\} = \{i, j, k\})$, i.e., $\|(p - p_i) + (p_j - p_k)\| = \|(p - p_i) - (p_j - p_k)\|$.

Some results





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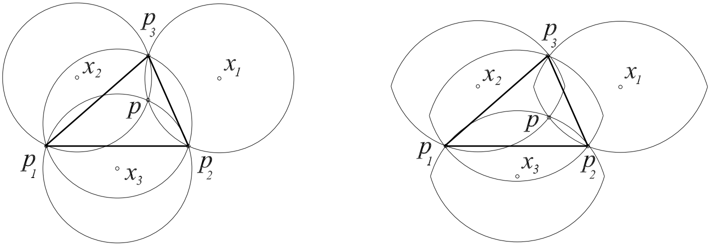

Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things. Isaac Newton

Theorem 4.2

In this situation, the quadruples $\{p, p_1, p_2, p_3\}$ *and* $\{o, x_1, x_2, x_3\}$ *both form C-orthocentric systems.*

Some results

Observation 4.1

Often statements which are true in $\mathbb{E}^2$ for circles of different sizes have Minkowskian analogues only for congruent circles.

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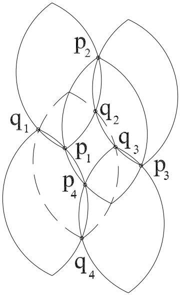
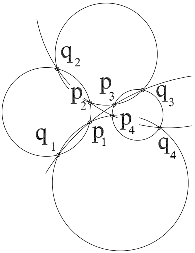
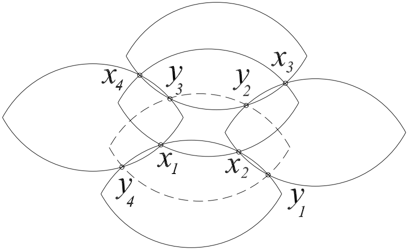




Miquel's theorem

Theorem 4.3

Let C be the unit circle in a strictly convex normed plane with $x_1, x_2, x_3, x_4 \in C$. If C_i are the four translates of C determined by pairs of neighboring points from $\{x_1, x_2, x_3, x_4\}$, then either there exists a translate of C passing through the four points y_i, where $y_i \in C_i \cap C_{i+1}$ $(C_5 = C_1)$ and $y_i \notin C$, or $y_i = x_i$ $(i = 1, 2, 3, 4)$.



Observation 4.2

Some theorems from $\mathbb{E}^2$ still hold in strictly convex normed planes, but many of them only "partially". E.g., the *Euler line* exists in triangles with unique circumcenter, and the *nine-point circle* of Feuerbach remains only as a *six-point circle*.

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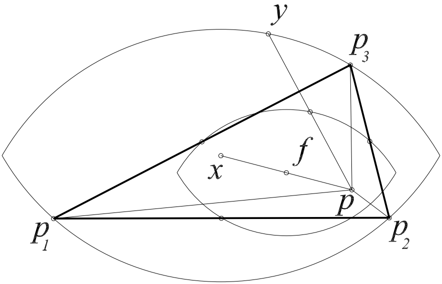
> *The one who understands geometry is able to understand everything in this world.* Galileo Galilei

Theorem 4.4

Let $p_1p_2p_3$ be a triangle in a strictly convex normed plane with circumcircle $x + \lambda C$ and C-orthocenter p. The circle $\frac{1}{2}(x+p) + \frac{1}{2}C$ passes through the midpoints of the sides of $p_1p_2p_3$ and the midpoints of $[p, p_i]$, $i = 1, 2, 3$.

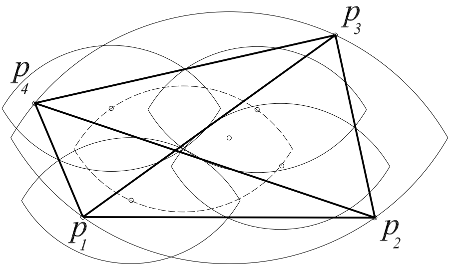


Remark 4.1

This *six-point circle* becomes a *(Feuerbach or) nine-point circle* (with $[pp_i] \cap [p_j p_k], \{1,2,3\} = \{i,j,k\}$, as additional three points) for *any* triangle iff $\mathbb{M}^2$ is Euclidean.

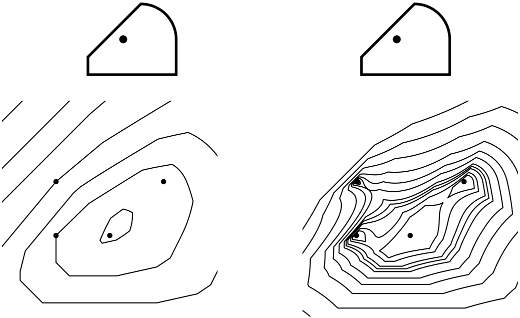
Theorem 4.5

Let p_1, p_2, p_3, p_4 be four pairwise distinct points in a strictly convex normed plane lying on $x + \lambda C$. Then the six-point circles of the four triangles obtainable from $\{p_1, p_2, p_3, p_4\}$ pass through a common point q, and their centers lie on $q + \frac{1}{2}\lambda C$ (the Feuerbach circle of the quadrilateral $p_1 p_2 p_3 p_4$).



5. Outlook: Generalized Minkowski spaces (gauges)

If the unit ball B is still a convex body with the origin in its interior, but the central symmetry of B is no longer demanded, then we have a *generalized Minkowski space* (with *convex distance function* or *gauge*).



New results of the Geometry group in Chemnitz refer also to the geometry of gauges!

Where there is matter, there is geometry! Johannes Kepler

Thank you!