Vertex index of symmetric convex bodies

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based on joint works with

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Motivation

Covering Conjecture (Hadwiger, 57; Gohberg-Markus, 60)

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Let K be a convex body in $\mathbb{R}^d$ with non-empty interior.

Def. 1. A point $p \in \mathbb{R}^d \setminus K$ illuminates a boundary point q of K if the ray emanating from p and passing through q intersects the interior of K (after the point q).

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Def. 2. A family of exterior points of K, $\{p_1, p_2, \dots, p_m\} \subset \mathbb{R}^d \setminus K$, illuminates K if each boundary point of K is illuminated by at least one of p_i's.

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Remark 2. Two conjectures above are equivalent (Boltyanski).

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This insures that far-away points of illumination are penalized.

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K. Bezdek posed the problem of finding the upper bound for the ill(K). He also provided some estimates and conjectured that for every symmetric body K

$$\mathrm{ill}(K) \geq 2d \quad \text{and} \quad \mathrm{ill}(B_2^d) = 2d^{3/2}$$

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$$\mathrm{cov}(K) = \inf \left\{ \sum_i \frac{1}{1-\lambda_i} \;\middle|\; K \subset \bigcup_i (x_i + \lambda_i K), 0 < \lambda_i < 1, x_i \in \mathbb{R}^d \right\}.$$

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Theorem (Swanepoel)

For every symmetric convex body K in $\mathbb{R}^d$ one has

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Recently, K. Bezdek and M. Khan have introduced a related notion – *covering index*.

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Def. Let K be a symmetric convex body in $\mathbb{R}^d$. We introduce the *vertex index* of K as follows:

$$\operatorname{vein}(K) = \inf \left\{ \sum_{i=1}^{m} \|p_i\|_K \;\Big|\; K \subset \operatorname{conv}\{p_i\}_{i \le m} \right\}.$$

Simple properties

Below K, L are symmetric convex bodies, T is an invertible linear operator, $d(\cdot, \cdot)$ denotes the Banach-Mazur distance, that is

$$d(K, L) = \inf \{\lambda > 0 \mid K \subset SL \subset \lambda K, \ S \ \text{ is an invertible linear operator}\}.$$

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Remark. Note that $\text{ill}(B_\infty^d) = 2^d$, while below we will see that $\text{vein}(K) \leq Cd^{3/2}$. It shows that $\text{ill}(K)$ is rather unstable, while Claim 2 shows that $\text{vein}(K)$ is stable.

Results

Theorem

For every symmetric convex body K in $\mathbb{R}^d$ one has

$$\frac{d^{3/2}}{\sqrt{2\pi e}\,\mathrm{ovr}(K)} \leq \mathrm{vein}(K) \leq 24\, d^{3/2}.$$

Here $\mathrm{ovr}(K)$ is the outer volume ratio of K, $\mathrm{ovr}(K) = \inf\left(\mathrm{Vol}(\mathcal{E})/\,\mathrm{Vol}(K)\right)^{1/d}$, where the infimum is taken over all ellipsoids $\mathcal{E} \supset K$ and $\mathrm{Vol}(\cdot)$ denotes the volume.

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1. the unit balls of ℓ_p, denoted by B_p^d:
for $p \ge 2$: $\mathrm{ovr}(B_p^d) \le C$; for $1 \le p \le 2$: $\mathrm{ovr}(B_p^d) \approx d^{1/p-1/2}$, $\mathrm{vein}(B_p^d) \approx d^{2-1/p}$

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3. some bodies with large ovr, e.g. $\mathrm{vein}(B_1^d) = 2d$ and $\mathrm{ovr}(B_1^d) \approx \sqrt{d}$.

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Thus

$$\mathrm{ovr}(P) \cdot \mathrm{vein}(P) \geq \frac{c\, d^2}{\sqrt{\ln(2d)}}.$$

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As a consequence, if $K \subset \mathbb{R}^2$, $L \subset \mathbb{R}^3$ are symmetric convex bodies, then

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2. The regular hexagon shows that the upper estimate 6 in the planar case is sharp.

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Remarks. 1. As an example of the octahedron shows, the lower estimates are sharp.
2. The regular hexagon shows that the upper estimate 6 in the planar case is sharp.
3. We do not know the best possible upper estimate in the 3-dimensional case.

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We show how the latter theorem implies the former one.

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Assume $m < 2d$. Since $K = -K \subset P$, we have $\|x\|_K \ge \|-x\|_P$ for every $x \in \mathbb{R}^d$. Therefore, applying our Theorem, we obtain

$$\sum_{i=1}^{m} \|p_i\|_K \ge \sum_{i=1}^{m} \|-p_i\|_P \ge \frac{m^2}{2k} = \frac{(d+k)^2}{2k} \ge 2d.$$

□

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Using Claim 2 (or just direct computations) we obtain

$$\mathrm{vein}(K) \le 3\sqrt{d}\,\mathrm{vein}(P) \le 24d^{3/2}.$$

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Finally, since B_2^d is the minimal volume ellipsoid for K and $\|\cdot\|_K \geq |\cdot|$, we have

$$\frac{1}{\text{ovr}(K)} = \left(\frac{\text{vol}\,(K)}{\text{vol}\,(B_2^d)} \right)^{1/d} \leq \left(\text{vol}\,(B_2^d)\right)^{1/d} \frac{1}{d} \sum_{i=1}^N \|p_i\|_K \leq \frac{\sqrt{2\pi e}}{d^{3/2}} \sum_{i=1}^N \|p_i\|_K .$$

Proof of “asymmetry" theorem.

Theorem. If $K = \mathrm{conv}\{x_i\}_{i \le m} \subset \mathbb{R}^d$ with $m = k + d \le 2d$ then

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Using $\langle f, x_i \rangle = \langle f, Te_i \rangle = \langle T^* f, e_i \rangle$, we obtain

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Therefore

$$\|z\|_{S^\circ} := \begin{cases} \sum_{j=1}^{m} \langle z, e_j \rangle & \text{if } \langle z, e_j \rangle \geq 0 \text{ for every } j \leq m, \\ \infty & \text{otherwise.} \end{cases}$$

Proof

Then $\|z\|_{PS^\circ} = \inf_{y\in L} \|z+y\|_{S^\circ}$

$$= \inf\left\{ \sum_{j=1}^{m} \langle z+y, e_j\rangle \;\middle|\; y\in L,\ \langle y, e_j\rangle \geq -\langle z, e_j\rangle \ \text{ for every } \ j\leq m \right\}.$$

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Then $y_{ij} \geq 0$ and $y_{ii} \geq 1$, and the matrix $\{y_{ij}\}$ has rank at most k.
For such matrices (**Lemma**) one has

$$A = \sum_{i=1}^{m} \sum_{j=1}^{m} y_{ij} - m \geq \frac{m(m-1)}{2k-1} \geq \frac{m^2}{2k}.$$

Proof of Lemma.

Lemma. Let $\Lambda = \{\lambda_{ij}\}$ be an $m \times m$ matrix of rank k with nonnegative entries such that $\lambda_{ii} \geq 1$ for every $i \leq m$. Then

$$\forall m \quad \sum_{i,j} \lambda_{ij} \geq 3m - 2k \qquad \text{and} \qquad \forall m \geq 2k \quad \sum_{i,j} \lambda_{ij} \geq m + \frac{m(m-1)}{2k-1}.$$

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Since Λ is of rank k, at least $m - k$ of eigenvalues of T are equal to -1

$$0 = \sum_{i=1}^{m} t_{ii} = \text{Trace} T = \sum_{i=1}^{m} \lambda_i,$$

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Since Λ is of rank k, at least $m - k$ of eigenvalues of T are equal to -1

$$0 = \sum_{i=1}^{m} t_{ii} = \mathrm{Trace} T = \sum_{i=1}^{m} \lambda_i,$$

Thus, using Weil's Theorem,

$$\sum_{i,j=1}^{m} t_{ij} \geq \sum_{i=1}^{m} |\lambda_i| \geq 2m - 2k.$$

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$$\sum_{i,j=1}^{m} \lambda_{ij} = m + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_{ij} = m + \binom{m-2}{l-2}^{-1} \sum_{\substack{\sigma\subset\{1,2,\ldots,m\}\\ |\sigma|=\ell}} \sum_{\substack{i,j\in\sigma\\ i\neq j}} \lambda_{ij}$$

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The choice $\ell = 2k$ completes the proof.

The non-symmetric case

Vertex index can be defined similarly (minimizing over all choices of the center):

$$\mathrm{vein}(K) = \inf \left\{ \sum_{i=1}^{m} \|p_i\|_{K-a} \;\middle|\; a \in K,\; K - a \subset \mathrm{conv}\{p_i\}_{i \le m} \right\}.$$

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This problem is closely related to approximation of convex bodies by polytopes (with small amount of verteces) in terms of Banach-Mazur distance.

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Lower bound for the Euclidean ball

The above proof gives $\mathrm{vein}(B_2^d) \geq d^{3/2}/\sqrt{2\pi e}$. Here we suggest another approach to the problem, which is of independent interest, and leads to the bound $d^{3/2}/\sqrt{3}$.

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Let $s_1 \geq s_2 \geq \ldots \geq s_d > 0$ be the singular values of S and let $\{w_i\}_{i \leq n}$, $\{z_i\}_{i \leq d}$ be orthonormal systems such that $S = \sum_{i=1}^{d} s_i w_i \otimes z_i$.

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Considering the extreme points of the section of the cube $B_\infty^N \cap E_m$ we observe that there exists a vector $y = \{y_i\}_{i \leq N} \in B_\infty^N \cap E_m$ such that the set $A = \{i \mid |y_i| = 1\}$ has cardinality at least $d + 1 - m$.

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WLOG we assume that $|A| = d+1-m$ (otherwise pass to a subset of A). Then

$$|(a\Lambda)^{-1}y| = \frac{1}{a}\sqrt{\sum_{i=1}^{N}\frac{y_i^2}{\lambda_i^2}} \geq \frac{1}{a}\sqrt{\sum_{i\in A}\frac{1}{\lambda_i^2}} \geq \frac{d+1-m}{a\sqrt{\sum_{i\in A}\lambda_i^2}} \geq \frac{d+1-m}{a\sqrt{\sum_{i=1}^N \lambda_i^2}} = \frac{d+1-m}{a}.$$

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This shows $s_m \geq (d+1-m)/a$ and implies

$$\frac{d^3}{3a^2} \leq \frac{1}{a^2}\sum_{m=1}^{d}(d+1-m)^2 \leq \sum_{m=1}^{d} s_m^2 = 1,$$

which proves the desired result.