Vertex-Transitive Polyhedra, Their Maps, and Quotients

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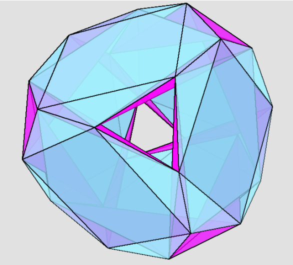
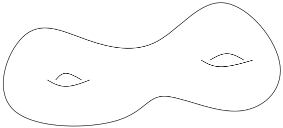
June 29, 2015

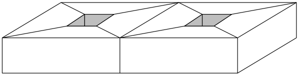


Polyhedra

Definition 1.1

A *polyhedron* P is a closed, connected, orientable surface embedded in $\mathbb{E}^3$ which is tiled by finitely many plane simple polygons in a face-to-face manner. The polygons are called *faces* of P, their vertices and edges are called the *vertices* and *edges* of P, respectively.





Note: Coplanar and non-convex faces are allowed.

Polyhedra, Polyhedral Maps, and Symmetry

A polyhedron P is the image of a *polyhedral map* M on a surface under a *polyhedral embedding* $f : M \to \mathbb{E}^3$. We call M the *underlying polyhedral map* of P. P is a *polyhedral realization* of M.

geometric *symmetry group* of a polyhedron: $G(P) \subset O(3)$
automorphism group of its underlying map M: $\Gamma(M)$
$G(P)$ is isomorphic to a subgroup of $\Gamma(M)$.

Definition 1.2

P is *(geometrically) vertex-transitive* if G acts transitively on the vertices of P.

Can we find all combinatorial types of geometrically vertex-transitive polyhedra (of higher genus $\mathfrak{g} \geq 2$)?

Vertex-Transitive Polyhedra of Higher Genus

- Grünbaum and Shephard (GS, 1984): vertex-transitive polyhedra of positive genus exist
- genus 1: two infinite two-parameter families based on prisms and antiprisms
- genus $3, 5, 7, 11, 19$: five examples, based on snub versions of Platonic solids
- the combinatorially regular Grünbaum polyhedron (Grünbaum (1999), Brehm, Wills) of genus 5
- most recent survey and a seventh example (of $\mathfrak{g} = 11$): Gévay, Schulte, Wills 2014 (GSW)

related concept: uniformity (vertex-transitivity with regular faces)

Vertex-Transitive Polyhedra – Examples of genus 11

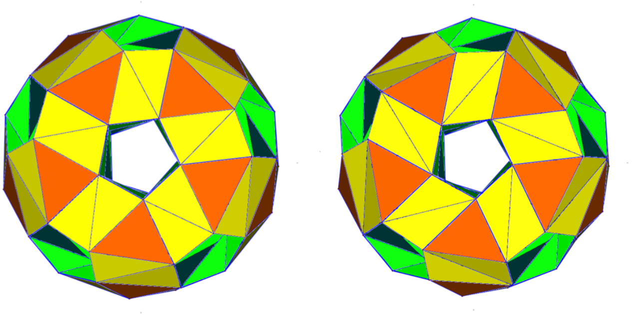
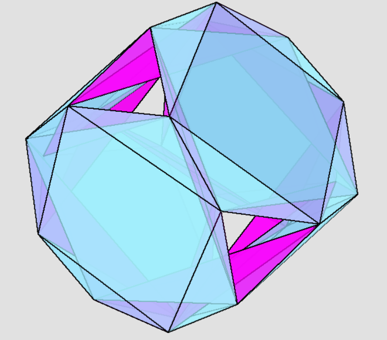
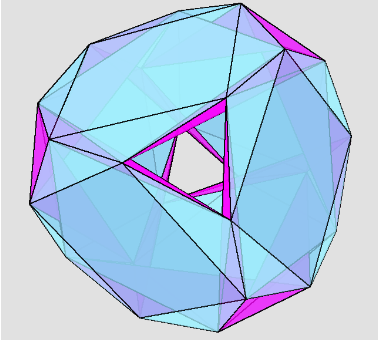


image source: Gévay, Schulte, Wills, *The regular Grünbaum polyhedron of genus* 5, Adv. Geom. 14(3) , 2014

A Vertex-Transitive Polyhedron of Genus 7 With Octahedral Symmetry

integer coordinates for the initial vertex: $v = (10, -4, 11)$





Example: Genus 5 with Octahedral Symmetry

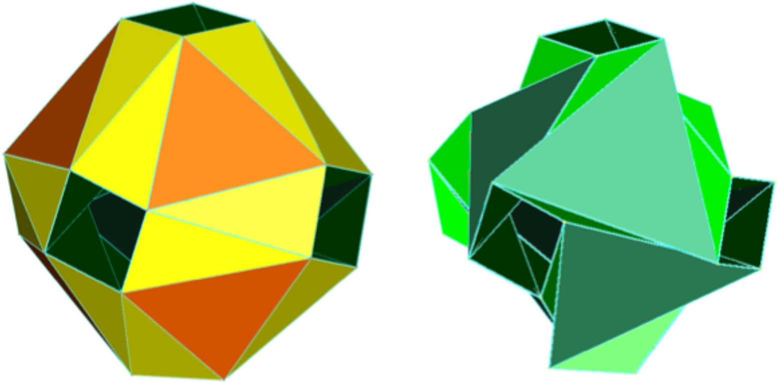


image source: GSW

The Combinatorially Regular Grünbaum Polyhedron

edge flips produce the Grünbaum polyhedron:

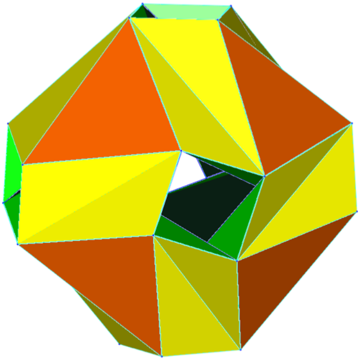


image source: GSW

Properties of Vertex-Transitive Polyhedra

- GS: All vertices lie on a sphere, and faces are convex polygons.
- GSW: The symmetry groups of vertex-transitive polyhedra for $g \geq 2$ are the rotation groups of the Platonic solids. There are finitely many such polyhedra in total (not just in each genus).
- Tucker (2014): The genus of an embedded (smooth) surface limits the possible geometric symmetries.
  - tetrahedral symmetry: $\mathfrak{g} \in L(6, 8) + \{0, 3, 5, 7\}$
  - octahedral symmetry: $\mathfrak{g} \in L(12, 16, 18) + \{0, 5, 7, 11, 13\}$
  - icosahedral symmetry: $\mathfrak{g} \in L(30, 40, 48) + \{0, 11, 19, 21, 29, 31, 37\}$

More Results

Note: For the rest of this talk, we assume higher genus in the range $\mathfrak{g} \geq 2$.

Theorem 1.1

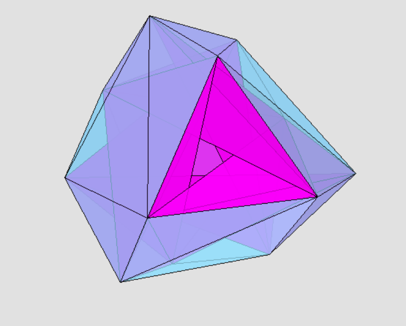
(L.) The geometric symmetry group $G(P)$ of a vertex-transitive polyhedron of genus $g \geq 2$ acts simply transitively on the vertices of P.

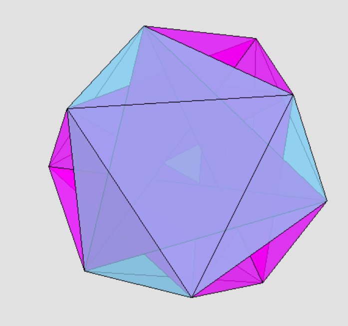
Theorem 1.2

(L.) There is only one combinatorial type of vertex-transitive polyhedron with $g \geq 2$ and tetrahedral symmetry. It has genus 3, is of type $\{3, 8\}$, and its convex hull is a snub tetrahedron.

The Vertex-Transitive Polyhedron of Genus 3 With Tetrahedral Symmetry

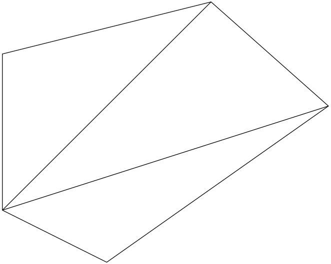
integer coordinates for the initial vertex: $v = (6, 1, 2)$



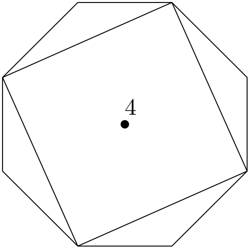


Maximal Triangulation

face with trivial stabilizer (*type* 1) example



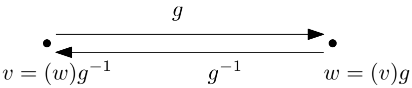
face with non-trivial stabilizer (*type* 2) example: stabilized by order 4 rotation



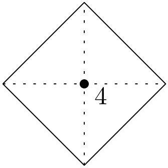
If a *maximally triangulated* map is not symmetrically realizable, then no map derived from it by making faces coplanar is.

Encoding Symmetry into the Underlying Maps

The darts (directed edges) can be labeled by uniquely determined group elements.

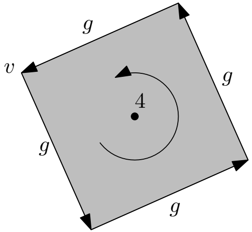


Not all group elements may appear. Labels are *core rotations* (rotations by the smallest nontrivial angle around an axis) for the Platonic rotation group G.



Decomposition of the Neighborhood of an Initial Vertex *v*

face orbit of type 2 (non-trivial stabilizer): single regular polygons

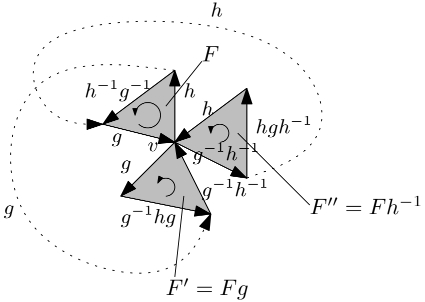


We call (g) the *orbit symbol* for an orbit of type 2. It encodes the angle (g, g).

Decomposition of the Neighborhood of an Initial Vertex v

face orbit of type 1 (with trivial stabilizer): triple of triangles

$g \neq 1$, $h \neq 1$
$gh \neq 1$



We call $(g, h, g^{-1}h^{-1})$ the *orbit symbol* for an orbit of type 1. It encodes *angles* (g, h), $(h, g^{-1}h^{-1})$, $(g^{-1}h^{-1}, g)$.

Candidate Maps

The underlying maps, when oriented, can be encoded succinctly by a list of *face orbit symbols*.

- reason: need information only at a designated *initial vertex*
- conversely, lists of orbit symbols are called *candidate map* if certain necessary conditions are met
- relation to branched coverings of surfaces, voltage graph constructions
- possible face orbit symbols can be determined geometrically by *pole figures*

First Approach

... enumerate candidate maps fulfilling the genus restrictions up to *geometric isomorphism*, then look for obstructions to symmetric realizability (as a geometric polyhedron with the encoded vertex-transitive symmetry).

tetrahedral symmetry:	3 suitable candidate maps of genus $g \geq 2$
octahedral symmetry:	860 suitable candidate maps of genus $g \geq 2$
icosahedral symmetry:	? suitable candidate maps

Octahedral Symmetry

There are precisely six maps of genus 5, and four maps of genus 7 in the list for octahedral symmetry; these are all the maps of small positive ($\mathfrak{g} \leq 10$) genus.

Theorem 2.1

(L.) In the genus range $0 < \mathfrak{g} \leq 10$*, there are precisely three maximally triangulated, vertex-transitive polyhedra with octahedral symmetry. Two are of genus* 5 *(the Grünbaum polyhedron and its relative), and one is of genus* 7*.*

Note that coplanarity of faces is impossible for the existing polyhedra.

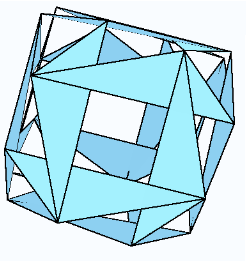
Why is Symmetric Realizability a Hard Problem?

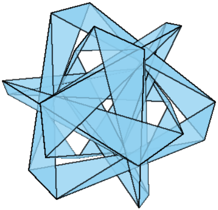
Symmetric Realizability Problem:

- difficult problem, in principle decidable
- theory of oriented matroids can simplify testing of necessary conditions

At a hands-on geometric level, you find that:

- for most of the face orbits, you can find a non-selfintersecting realization
- there are too many maps to consider





A Second Approach

Let S be the surface supporting P and consider $p : S \to S/G$. The only points at which p is not an ordinary covering projection are called *branch points*. Around these points, p looks like the complex exponential function $z \mapsto z^n$.

face structure	$\to$	cell structure of the quotient
face orbits of type 2	$\to$	branch points of higher *order*
orbit of edges fixed by half-turns	$\to$	semi-edges with branch points of *order* 2

The Euler characteristics of S and S/G are related by the Riemann-Hurwitz equation.

Orbifolds of E^3

Consider the quotient map $q : E^3 \to E^3/G$. On each concentric sphere $S_r = S(o, r)$ in E^3 the map q is a regular branched covering of a spherical quotient.

The set X of rotation axes gets taken to a set Y of rays in E^3/G (made up of the branch points on the quotients of the concentric spheres).

It is possible to recover the original space E^3 by a voltage graph construction: assign generators u, v of G to generators a, b of $\pi(E^3/G - Y)$ for some fixed base point.

Orbifolds of E^3

G a group of rotations, $Y \subset E^3/G$ consists of:

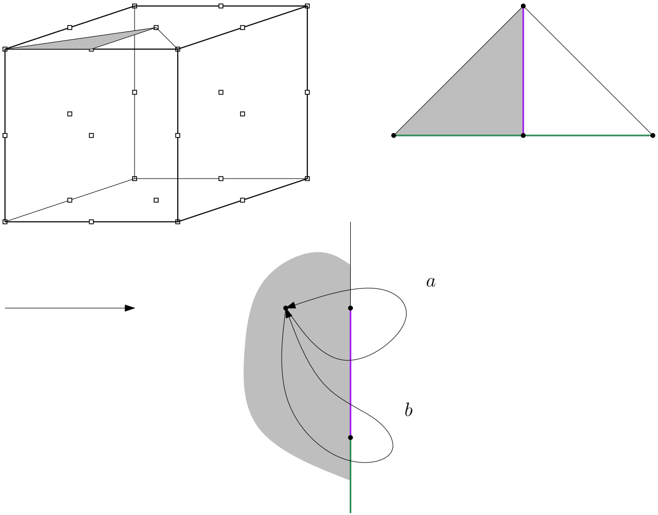
- a 2-fold and two 3-fold rays from the origin for tetrahedral symmetry (A_4)
- a 2-fold, a 3-fold, and a 4-fold ray from the origin for cubical symmetry (S_4)
- a 2-fold, a 3-fold, and a 5-fold ray from the origin for icosahedral symmetry (A_5)

The Quotient Surface is Embedded in E^3/G

Let $f : M \to P \subset E^3$ be a symmetric polyhedral embedding of a map M into E^3 and consider $p : M \to M/G = M_q$.
Let $B = g^{-1}(Y)$. There is an embedding $g : M_q \to E^3/G$ completing the commutative diagram:

$$\begin{array}{ccc} (M, p^{-1}[B]) & \xrightarrow{f} & (E^3, X) \\ \downarrow p & & \downarrow q \\ (M_q, B) & \xrightarrow{g} & (E^3/G, Y) \end{array}$$

Edges in the Quotient



Faces of Type 1 in the Quotient

- there are only 21 possible words (and their inverses) in $\langle a, b \rangle$ to which an edge / semiedge could be sent
- there can be at most 13 kinds of edges in a map
- for a face orbit of type 1, elements associated to boundary walk need to multiply to the identity (no axis pierces a triangle)

Theorem

(L.) There are less than 70 vertex-transitive polyhedra of higher genus for octahedral symmetry. The genera are in $\{5, 7, 12, 13, 17, 19, 24\}$.

Thank you!

Questions?