Robustness of convex solids

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Static equilibrium: when the body rests under gravity on a horizontal plane.

QUESTION

If we want to change the number of static equilibria of a given convex body by a suitable truncation, how large truncation do we need to make? (Largeness is measured, say, as the volume of the truncated part, relative to the full volume of the body)

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downward robustness

FIGURE: Beach with pebbles

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$$
\begin{array}{ccc}
 & \text{robustness} & \\
\swarrow & & \searrow \\
\text{internal} & & \text{external}
\end{array}
$$

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DEFINITION

For any $K \in \mathcal{K}_2$ and $p \in \operatorname{int} K$, $q \in \operatorname{bd} K$ is an *equilibrium point* of K with respect to p, if the line passing through q and perpendicular to $q - p$, supports K.

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REMARK

If $\operatorname{bd} K$ is smooth at q, it is equivalent to saying that q is a critical point of the Euclidean distance function $z \mapsto |z - p|$, $z \in \operatorname{bd} K$.

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Remark

If $K \in \mathcal{O}_2$ or $K \in \mathcal{P}_2$, then this definition reduces to the usual concept of nondegeneracy in these classes.

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- Poincaré-Hopf Theorem: the numbers of the stable and unstable equilibrium points of any $K \in \mathcal{K}_2$ are equal.
- These two types of points form an alternating sequence in $\mathrm{bd}\, K$.

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Notation:

- $\{S\}$: family of convex bodies in $\mathcal{K}_2$, with S stable points with respect to their centres of gravity.
- $\{S, U\}$: family of convex bodies in $\mathcal{P}_3 \cup \mathcal{O}_3$, with S stable and U unstable points with respect to their centres.

For $K \in \{S\}$, set

$$\mathcal{F}_{<}(K) = \{K' \subset K : K' \in \{S'\} \text{ for some } S' < S\}.$$

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DEFINITION

Let $K \in \{S\}$. Then we define the downward robustness (or simply *robustness*) of K as the quantity

$$\rho(K) = \frac{\inf\{\mathrm{Area}(K \setminus K') : K' \in \mathcal{F}_{<}(K)\}}{\mathrm{Area}(K)}.$$

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In $\mathbb{R}^3$, we may define $\rho(K)$ in an analogous way. Set $\rho_S = \sup\{\rho(K) : K \in \{S\}\}$, $\rho_{S,U} = \sup\{\rho(K) : K \in \{S, U\}\}$.

External robustness

For $K \in \mathcal{K}_2$ with S stable points with respect to $p \in \operatorname{int} K$, set

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In $\mathbb{R}^3$, we may define $\rho^{ex}(K,p)$ in an analogous way. Set
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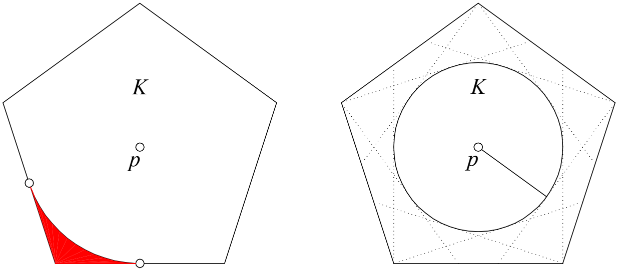

FIGURE: External and internal robustness

Results: external robustness in the plane

THEOREM

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COROLLARY

For any $S \geq 3$, we have $\rho^{ex}_S = \frac{\tan\frac{\pi}{S} - \frac{\pi}{S}}{S\tan\frac{\pi}{S}}$, and the convex bodies in $\{S\}$ with maximal external robustness are the regular S-gons.

Results: internal robustness in the plane

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COROLLARY

For any $S \geq 3$, we have $\rho^{in}_S = \frac{1}{2S}$, and the plane convex bodies $K \in \{S\}$ with maximal internal robustness with respect to their centres of gravity are the regular S-gons.

Results: internal robustness in 3-space

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$$\rho^{in}(P', q) \leq \rho^{in}(P, o),$$

with equality if, and only if P' is a similar copy of P, with q as its centre.

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REMARK

If, say, P and P' have the same number of stable points, unstable points and edges, then they have the same number of faces, vertices and saddle points.

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Results: (full) robustness in 3-space

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THEOREM

We have $\rho_{12} = \rho_{21} = \rho_{22} = 1$.

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We may define *partial robustness*, i.e. the (relative) volume of a truncation necessary to reduce *either S or U*, (the numbers of stable and unstable points, respectively. We call this *S*-robustness and *U*-robustness, denoted by $\rho^s(K), \rho^u(K)$, respectively.

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Clearly, we have $\rho(K) = \min\{\rho^s(K), \rho^u(K)\}$, and also $\rho^s_{1,n} = \rho^u_{n,1} = 1$ for any $n > 2$.

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Theorem

If $n > 2$, then $\rho^s_{2,n} = \rho^u_{n,2} = 1$.

RESULTS: PARTIAL ROBUSTNESS IN 3-SPACE

	U=1	U=2	U=3	U=4
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REMARK

In coastal regions the percentage of pebbles in classes $\{1, n\}, \{n, 1\}$ was found to be below 0.1%.

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Remark

For every $S \geq 3$, the maximum of $\rho^{in}(K, p)$ over $\mathcal{K}_2$ can be approached by regions with smooth boundaries as well; or in other words, $\frac{1}{2S} = \sup\{\rho^{in}(K, p) : K \in \mathcal{O}_2, p \in \operatorname{int} K\}$.

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The theorem about the internal robustness of platonic solids is valid also for downward external robustness, and for downward full robustness.

We call the equilibrium classes containing platonic solids (classes $\{4,4\}, \{6,8\}, \{8,6\}, \{20,12\}$ and $\{12,20\}$) platonic classes.

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PROBLEM

Prove or disprove that in the platonic classes platonic solids have maximal downward full robustness.

Questions; alternative definition for robustness

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If $i \geq k$ and $j \geq l$ then $\rho_{i,j} \leq \rho_{k,l}$.

Questions; alternative definition for robustness

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Alternative definition: we may permit only truncations by planes. Advantage:

- numerical experiments are feasible;
- this is the only truncation such that both resulting pieces are convex.

SUMMARY

robustness

$\swarrow$ $\downarrow$ $\downarrow$ $\searrow$

internal external full partial

Average robustness

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The average truncation of K is the limit of the nth order average truncations, if it exists.

AVERAGE ROBUSTNESS

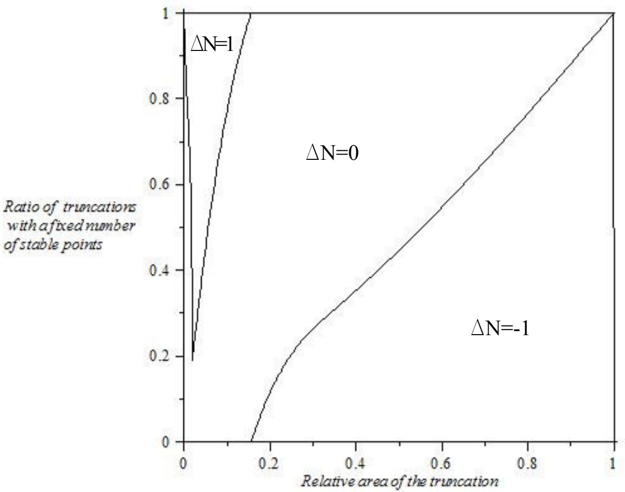

FIGURE: Truncations of a unit square with one line.

And now . . .

The End