Maximality properties of rational lattice-free polyhedra

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Joint work with Gennadiy Averkov and Stefan Weltge

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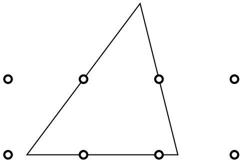

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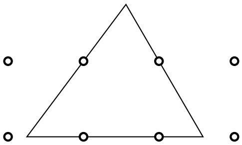

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- *For fixed d: only finitely many $\mathbb{Z}^d$-maximal lattice-free elements of $\mathcal{P}(\mathbb{Z}^d)$ up to unimodular equivalence.*

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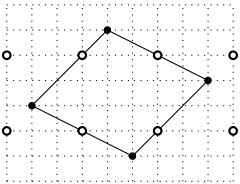

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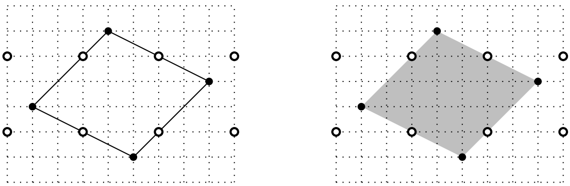

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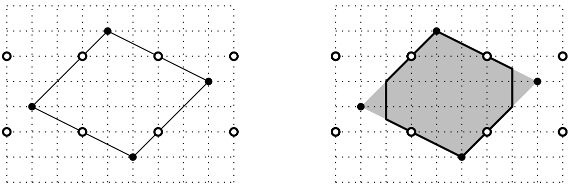

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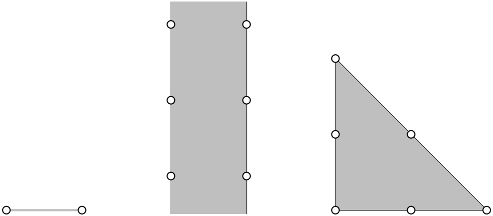

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Theorem (Averkov & K. & Weltge (2015+))

Let $P \in \mathcal{P}(\mathbb{Z}^3)$ be lattice-free. Then P is $\mathbb{Z}^3$-maximal if and only if P is $\mathbb{R}^3$-maximal.

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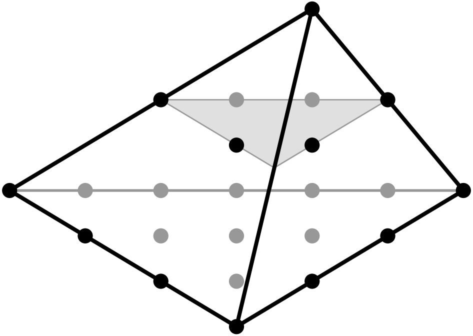

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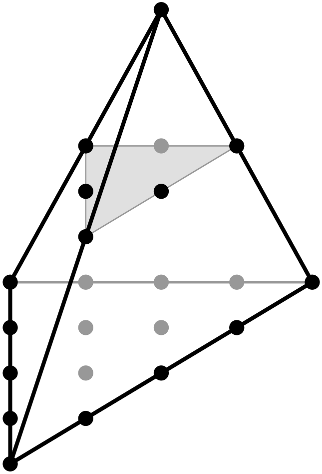

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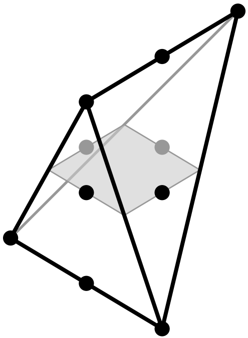

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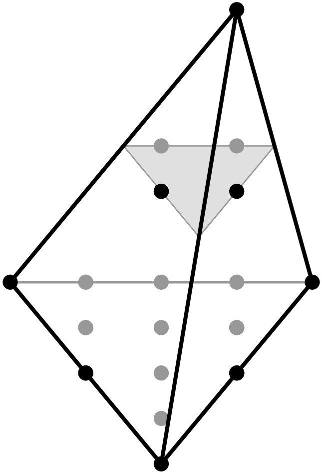

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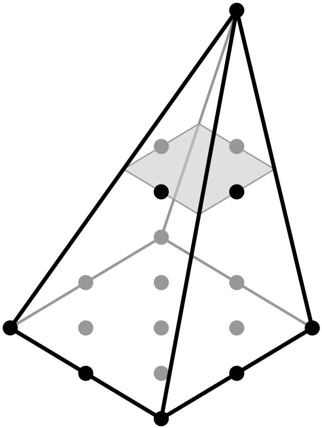

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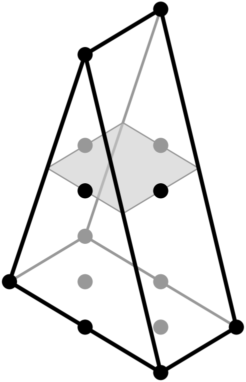

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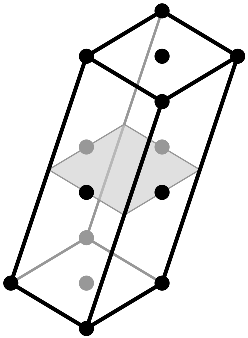

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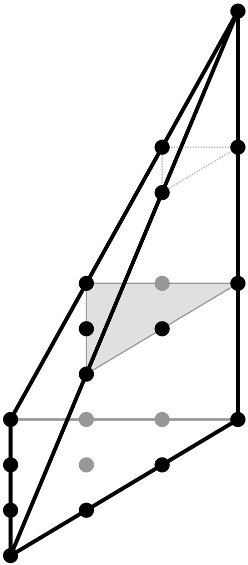

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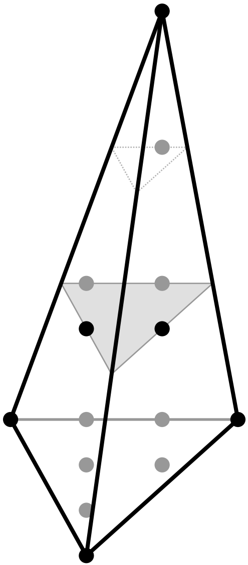

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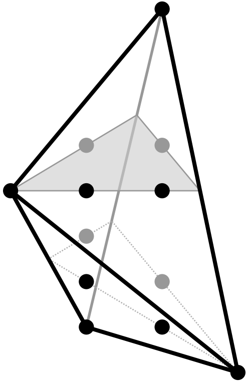

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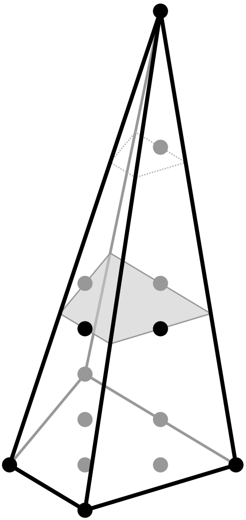

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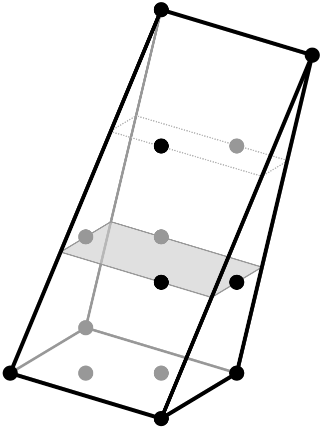

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Proof idea(cont.)

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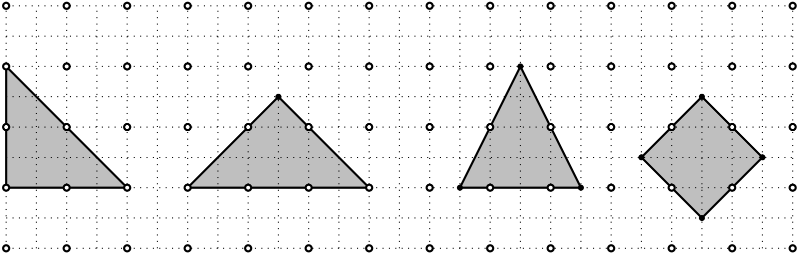
Construct P from P_0. Result: enumeration of all seven $\mathbb{Z}^3$-maximal polytopes in $\mathcal{P}(\mathbb{Z}^3)$ with lattice width two; all of which are $\mathbb{R}^3$-maximal.

$\mathbb{Z}^2$-maximal polytopes in $\mathcal{P}(\frac{1}{2}\mathbb{Z}^2)$



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We derive an algorithm to find all $\mathbb{Z}^3$-maximal polytopes in $\mathcal{P}(\mathbb{Z}^3)$ with lattice width three or higher and check them for $\mathbb{R}^3$-maximality to complete the proof.

Theorem (Averkov & K. & Weltge (2015+))

Let $d, s \in \mathbb{N}$ and let $P \in \mathcal{P}(\frac{1}{s}\mathbb{Z}^d)$ be lattice-free. Then if $d = 1$ or $d = 2, s \in \{1, 2\}$ or $d = 3, s = 1$,

$$P \text{ is } \mathbb{Z}^d\text{-maximal if and only if } P \text{ is } \mathbb{R}^d\text{-maximal.}$$

Furthermore, for all other pairs d, s, this equivalence does not hold.

Thank you for your attention!

Köszönöm!