# Quantitative covering of convex bodies

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# Background

Let $\mathbb{E}^d$ denote the $d$-dimensional Euclidean space. A *$d$-dimensional convex body* $K$ is a compact convex subset of $\mathbb{E}^d$ with nonempty interior.

**Conjecture 1 (Hadwiger Covering Conjecture (1960))**

*$K$ can be covered by $2^d$ of its smaller positive homothets and $2^d$ homothets are needed only if $K$ is an affine $d$-cube.*

The **illumination number** $I(K)$ of $K$ is the smallest $n$ for which the boundary of $K$ can be illuminated by $n$ points/directions.

Boltyanski (1960) showed that $I(K) = n$ if and only if the smallest number of smaller positive homothets of $K$ that cover $K$ is $n$.

**Conjecture 2 (Illumination Conjecture (1960))**

*$I(K) \leq 2^d$, and $I(K) = 2^d$ only if $K$ is an affine $d$-cube.*

**Question:** How 'economically' can we cover $K$ by a few small homothets?

Swanepoel (2005) defined the **covering parameter** of a $d$-dimensional convex body.

$$C(K) = \inf \left\{ \sum_i \frac{1}{1-\lambda_i} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{E}^d \right\}.$$

- Large homothets are penalized.
- $I(K) < C(K)$.
- $C(K) = O(2^d d^2 \ln d)$, if $K$ is $o$-symmetric.
- $C(K) = O(4^d d^{3/2} \ln d)$, in general.
- If $K$ is $o$-symmetric,
$$\text{ill}(K) \leq 2C(K),$$
where $\text{ill}(K)$ is the **illumination parameter** of $K$ [Bezdek (1992)].
- Let $C^d$ denote a $d$-dimensional cube, then $C(C^d) = 2^{d+1}$.

Denote by $\mathcal{K}^d$ the (compact) space of $d$-dimensional convex bodies under the (multiplicative) Banach-Mazur distance:

$$d_{BM}(K, L) = \inf \{\delta \geq 1 : a \in K, b \in L, L - b \subseteq T(K - a) \subseteq \delta(L - b)\},$$

where the infimum is taken over all invertible linear operators $T : \mathbb{E}^d \longrightarrow \mathbb{E}^d$.

Define $\gamma_m(K)$ to be the **minimal homothety ratio** required to cover $K$ by $m$ positive homothets.

$$\gamma_m(K) = \inf \left\{ \lambda > 0 : K \subseteq \bigcup_{i=1}^{m} (\lambda K + t_i), t_i \in \mathbb{E}^d, i = 1, \ldots, m \right\}.$$

- Originally, introduced by Lassak (1986).
- Zong (2010) reintroduced it as a functional on $\mathcal{K}^d$ and proved it to be uniformly continuous.
- In fact, $\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L)$, for any $K, L \in \mathcal{K}^d$. [B-K (2015)]

# The covering index

**Definition 3**

Let $K \in \mathcal{K}^d$. We define the **covering index** of $K$ as

$$\mathrm{coin}(K) = \inf \left\{ \frac{m}{1-\gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N} \right\}.$$

Intuitively, $\mathrm{coin}(K)$ measures how well $K$ can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio not exceeding $1/2$.

Results on covering index appear in

K. Bezdek and M. A. Khan, The covering index of convex bodies, arXiv:1503.03111v3 [math.MG] (16 June, 2015).

## Why $\gamma_m(K) \leq 1/2$?

1) Rogers (1963), Verger-Gaugry (2005), O'Rourke (2012) and others investigated the minimum number of homothets of ratio $1/2$ or less needed to cover a $d$-dimensional ball.

2) Easier to find exact values (for infinitely many convex bodies), estimates and optimizers.

3) Define

$$f_m(K) = \begin{cases} \dfrac{m}{1-\gamma_m(K)}, & \text{if } \gamma_m(K) \leq 1/2, \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\operatorname{coin}(K) = \inf\{f_m(K) : m \in \mathbb{N}\}$.

For any $K, L \in \mathcal{K}^d$ and $m \in \mathbb{N}$ such that $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$,

$$f_m(K) \leq d_{BM}(K,L) f_m(L), \qquad f_m(K) \geq \frac{d_{BM}(K,L)}{2d_{BM}(K,L)-1} f_m(L).$$

The above relations don't work without restricting the homothety ratios.

## Proposition 4 (Relation with other quantities)

*For any o-symmetric d-dimensional convex body K,*

$$\mathrm{vein}(K) \le \mathrm{ill}(K) \le 2C(K) \le 2\,\mathrm{coin}(K),$$

*and in general*

$$I(K) < C(K) \le \mathrm{coin}(K).$$

Here vein($K$) denotes the the **vertex index** [Bezdek, Litvak (2007)] of the $o$-symmetric convex body $K$.

## Proposition 5 (Rogers-type bounds)

*Given $K \in \mathcal{K}^d$, $d \ge 2$, we have*

$$\mathrm{coin}(K) < \begin{cases} 2^{2d+1} d(\ln d + \ln\ln d + 5) = O(4^d d \ln d), & K \text{ } o\text{-symmetric}, \\ 2^{d+1}\binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(8^d d \ln d), & \text{otherwise}. \end{cases}$$

## Lemma 6 (Monotonicity)

*Let $j < m$ be positive integers. Then for any d-dimensional convex body $K$ the inequality $f_m(K) < f_j(K)$ implies $m < f_j(K)$.*

This shows that the covering index of any convex body can be obtained by calculating a finite minimum.

In particular, if $f_j(K) < \infty$ for some $j$, then

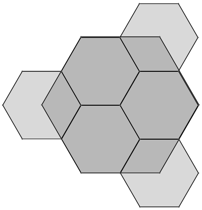
$$\operatorname{coin}(K) = \min\{f_m(K) : m < f_j(K)\}.$$

**Example**

An affine regular convex hexagon $H$ can be covered by 6 (and no fewer) half-sized homothets. Thus $\operatorname{coin}(H) \leq f_6(H) \leq 12$ and

$$\operatorname{coin}(H) = \inf\{f_m(H) : m < 12\} \leq 12.$$

(In fact, $\operatorname{coin}(H) = 12$.)



For any $K, L \in \mathcal{K}^d$, let $N(K, L)$ denote the **covering number** of $K$ by $L$, i.e., the minimum number of translates of $L$ needed to cover $K$.

$$\operatorname{coin}(K) = \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \leq \frac{1}{2}, m \in \mathbb{N}\right\} = \inf_{\lambda \leq \frac{1}{2}} \frac{N(K, \lambda K)}{1-\lambda}.$$

## Definition 7

We say that a convex body $K \in \mathcal{K}^d$ is **tightly covered** if for any $0 < \lambda < 1$, $K$ contains at least $N(K, \lambda K)$ points no two of which belong to the same homothet of $K$ with homothety ratio $\lambda$.

- The line segment $\ell \in \mathcal{K}^1$ is tightly covered.
- Any finite direct vector sum of tightly covered convex bodies is tightly covered.
- For $d \geq 2$, the $d$-dimensional cube $C^d$ is tightly covered.
- Not all convex bodies are tightly covered (e.g., the circle).

## Theorem 8 (Direct vector sums)

*(i) Let* $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ *be a decomposition of* $\mathbb{E}^d$ *into the direct vector sum of its linear subspaces* $\mathbb{L}_i$ *and let* $K_i \subseteq \mathbb{L}_i$ *be convex bodies, then*

$$\max_{1\le i\le n} \operatorname{coin}(K_i) \le \operatorname{coin}(K_1 \oplus \cdots \oplus K_n) \le \inf_{\lambda \le \frac{1}{2}} \frac{\prod_{i=1}^n N(K_i, \lambda K_i)}{1-\lambda} < \prod_{i=1}^n \operatorname{coin}(K_i).$$

*(ii) If in addition, any* $n-1$ *of the* $K_i'$*s are tightly covered, then*

$$\max_{1\le i\le n} \operatorname{coin}(K_i) \le \operatorname{coin}(K_1 \oplus \cdots \oplus K_n) = \inf_{\lambda \le \frac{1}{2}} \frac{\prod_{i=1}^n N(K_i, \lambda K_i)}{1-\lambda} < \prod_{i=1}^n \operatorname{coin}(K_i).$$

*(iii) For any* $(d+1)$*-dimensional 1-codimensional cylinder* $K \oplus \ell$*,*

$$\operatorname{coin}(K \oplus \ell) = 4N_{1/2}(K).$$

Let $\Delta^d$, $B^d$ and $C^d$ denote the $d$-simplex, $d$-dimensional ball and $d$-dimensional cube, respectively.

## Theorem 9 (Minkowski sums)

*Let* $K_1, \ldots, K_n \in \mathcal{K}^d$. *Then*

$$? \leq \operatorname{coin}(K_1 + \cdots + K_n) \leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N(K_i, \lambda K_i)}{1-\lambda} < \prod_{i=1}^n \operatorname{coin}(K_i).$$

## Theorem 10 (Continuity)

*Let $d$ and $m$ be any positive integers, $\mathcal{K}_m^d := \left\{K \in \mathcal{K}^d : \gamma_m(K) \le 1/2\right\}$ and $\mathcal{K}^{d*} := \left\{K \in \mathcal{K}^d : \gamma_m(K) \neq 1/2, m \in \mathbb{N}\right\}$.*
*(i) For any $K, L \in \mathcal{K}_m^d$,*

$$f_m(K) \le d_{BM}(K,L) f_m(L), \qquad f_m(K) \ge \frac{d_{BM}(K,L)}{2d_{BM}(K,L)-1} f_m(L). \tag{1}$$

*(ii) The functional $f_m : \mathcal{K}_m^d \longrightarrow \mathbb{R}$ is Lipschitz continuous. On the other hand, $f_m : \mathcal{K}^d \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.*
*(iii) Define $I_K = \{i : \gamma_i(K) \le 1/2\} = \{i : K \in \mathcal{K}_i^d\}$, for any $d$-dimensional convex body $K$. If $I_L \subseteq I_K$, for some $K, L \in \mathcal{K}^d$, then*

$$\operatorname{coin}(K) \le \frac{2d_{BM}(K,L)-1}{d_{BM}(K,L)} \operatorname{coin}(L) \le d_{BM}(K,L) \operatorname{coin}(L). \tag{2}$$

*(iv) The functional $\operatorname{coin} : \mathcal{K}^d \longrightarrow \mathbb{R}$ is lower semicontinuous.*
*(v) The functional $\operatorname{coin} : \mathcal{K}^{d*} \longrightarrow \mathbb{R}$ is continuous.*

## Theorem 11 (Optimizers)

*(i) For any* $K \in \mathcal{K}^d$, $\text{coin}(C^d) = 2^{d+1} \leq \text{coin}(K)$ *and so d-cubes minimize the covering index in all dimensions.*

*(ii) If* $K$ *is a planar convex body then* $\text{coin}(K) \leq \text{coin}(B^2) = 14$.

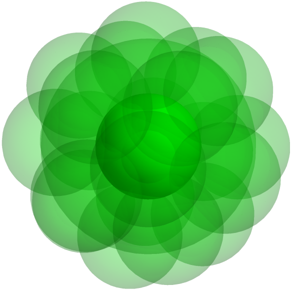
*(iii) If* $K \oplus \ell$ *is a 1-codimensional cylinder in* $\mathcal{K}^3$, *then* $\text{coin}(K \oplus \ell) \leq \text{coin}(B^2 \oplus \ell) = 28$.

Verger-Gaugry (2005) showed that in any dimension $d \geq 2$ one can cover a ball of radius $1/2 < r \leq 1$ with

$$O((2r)^{d-1} d^{3/2} \ln d)$$

balls of radius $1/2$. Thus

$$\text{coin}(B^d) = O(2^d d^{3/2} \ln d).$$



$\text{coin}(B^3) \leq 41.53398\ldots.$

## Open Problem

Prove that for any $d$-dimensional convex body $K$, $\text{coin}(K) \leq \text{coin}(B^d)$ holds.

An affirmative answer would immensely improve the known general upper bound on the illumination number from $O(4^d d \ln d)$ to $O(2^d d^{3/2} \log d)$.

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The covering index can be generalized in the following natural way.

**Definition 12**

Let $K \in \mathcal{K}^d$ and $0 < \lambda < 1$. We define the $\lambda$-**covering index** of $K$ as

$$\operatorname{coin}_\lambda(K) = \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \le \lambda, m \in \mathbb{N}\right\}.$$

Intuitively, $\operatorname{coin}_\lambda(K)$ measures how $K$ can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio not exceeding $\lambda$.

Also

$$\operatorname{coin}(K) = \operatorname{coin}_{1/2}(K).$$

What happens as $\lambda$ becomes large?

**Definition 13**

Let $K \in \mathcal{K}^d$. We define the **covering limit** of $K$ as

$$\text{colim}(K) = \lim_{\lambda \to 1^-} \text{coin}_\lambda(K).$$

Observe that

$$\text{colim}(K) = \inf\left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) < 1, m \in \mathbb{N} \right\}.$$

- Let $0 < \alpha < 1/2$ and $1/2 < \beta < 1$, then for any $o$-symmetric $d$-dimensional convex body $K$,

  $$\mathrm{ill}(K) \leq 2C(K) \leq 2\,\mathrm{colim}(K) \leq 2\,\mathrm{coin}_\beta(K) \leq 2\,\mathrm{coin}(K) \leq 2\,\mathrm{coin}_\alpha(K),$$

  and in general

  $$I(K) < C(K) \leq \mathrm{colim}(K) \leq \mathrm{coin}_\beta(K) \leq \mathrm{coin}(K) \leq \mathrm{coin}_\alpha(K).$$

- Several properties of the covering index such as **monotonicity**, **direct vector sum compatibility**, **Minkowski sum compatibility** hold for $\lambda$-covering indices and the covering limit.
- However, there are problems with **continuity**, determining **exact values** and finding **optimizers**.
- If $K$ is a planar convex body, then $\mathrm{colim}(K) \geq \mathrm{colim}(C^2) = 8$.
- K. Bezdek and M. A. Khan, Quantitative covering of convex bodies, (*preprint*).

The following table can be extended indefinitely by including coin- and colim-values (or estimates) for direct vector sums and Minkowski sums of convex bodies appearing in the table.

| $K$ | $m$ | $\gamma_m(K)$ | $\mathrm{coin}(K)$ | $m$ | $\gamma_m(K)$ | $\mathrm{colim}(K)$ |
|---|---|---|---|---|---|---|
| $\ell$ | 2 | $1/2$ | 4 | 2 | $1/2$ | 4 |
| $H$ | 6 | $1/2$ | 12 | 3 | $2/3$ | 9 |
| $\Delta^2$ | 6 | $1/2$ | 12 | 3 | $2/3$ | 9 |
| $B^2$ | 7 | $1/2$ | 14 | 5 | $0.609\ldots$ | $12.800\ldots$ |
| $B^3$ | $\geq 21$ | $\leq 0.49\ldots$ | $\leq 41.53\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $B^d$ | $O(2^d d^{3/2} \ln d)$ | $\leq 1/2$ | $O(2^d d^{3/2} \ln d)$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $C^d$ | $2^d$ | $1/2$ | $2^{d+1}$ | $2^d$ | $1/2$ | $2^{d+1}$ |
| $\Delta^d$ | $\ldots$ | $\ldots$ | $\ldots$ | $\geq d+1$ | $\leq \frac{d}{d+1}$ | $\leq (d+1)^2$ |
| $H \oplus \ell$ | 12 | $1/2$ | 24 | 12 | $1/2$ | 24 |
| $\Delta^2 \oplus \ell$ | 12 | $1/2$ | 24 | 12 | $1/2$ | 24 |
| $B^2 \oplus \ell$ | 14 | $1/2$ | 28 | 14 | $1/2$ | 28 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 1 : Known values (or estimates) of $\mathrm{coin}(\cdot)$ and $\mathrm{colim}(\cdot)$ together with the corresponding $m$ and $\gamma_m(\cdot)$.

**Classical covering numbers:**

$N(K, L) =$ minimum number of translates of $L$ needed to cover $K$.

$\overline{N}(K, L) =$ minimum number of translates of $L$ by points in $K$ needed to cover $K$.

**Weighted analogues:** [Artstein-Avidan, Raz (2011)]

A set $\{(\omega_i, x_i) : \omega_i > 0, x_i \in K\}_{i=1}^{n}$ of $n \in \mathbb{N}$ pairs is said to be an **internal weighted cover** of $K$ by $L$ if for all $x \in K$,

$$\sum_{i=1}^{n} \omega_i \mathbb{1}_{L+x_i}(x) \geq 1.$$

$\overline{N}_\omega(K, L) =$ infimal $\sum_{i=1}^{n} \omega_i$ over all internal weighted covers of $K$ by $L$.

If we remove the restriction of translating $L$ by points in $K$, we obtain the weighted covering number $N_\omega(K, L)$ [Artstein-Avidan, Slomka (2013)].

Warning:

The quantity $\overline{N}_\omega(K, L)$ and the results obtained in [Artstein-Avidan, Raz (2011)] work well when $L$ is $o$-symmetric but not in general.

The weighted covering number $N_\omega(K, L)$ works for all convex bodies. But not much is known about it.

Let $\mathcal{C}^d$ denote the subspace of $d$-dimensional $o$-symmetric convex bodies.

## Definition 14

Let $K \in \mathcal{K}^d$. We define the **fractional covering index** of $K$ as

$$\operatorname{fcoin}(K) = \inf_{\lambda \leq 1/2} \frac{N_\omega(K, \lambda K)}{1 - \lambda}.$$

Let $K \in \mathcal{C}^d$. We define the **internal fractional covering index** of $K$ as

$$\overline{\operatorname{fcoin}}(K) = \inf_{\lambda \leq 1/2} \frac{\overline{N}_\omega(K, \lambda K)}{1 - \lambda}.$$

Clearly, $\operatorname{fcoin}(K) \leq \overline{\operatorname{fcoin}}(K)$, for any $K \in \mathcal{C}^d$ and $\operatorname{fcoin}(K) \leq \operatorname{coin}(K)$, for any $K \in \mathcal{K}^d$.

The quantities $\operatorname{fcoin}_\lambda(K)$, $\overline{\operatorname{fcoin}}_\lambda(K)$, $\operatorname{fcolim}(K)$ and $\overline{\operatorname{fcolim}}(K)$ can be defined analogously.

## Theorem 15

*Let $K \in \mathcal{C}^d$. If for all $0 < \lambda < 1$, $\overline{N}_\omega(K, \lambda K) = \overline{N}(K, \lambda K)$, then for any $L \in \mathcal{C}^{d_0}$, $d_0 \in \mathbb{N}$ the body $K \oplus L$ is tightly covered.*

## Theorem 16

*Let $K \in \mathcal{K}^d$. If $K$ is tightly covered then for all $0 < \lambda < 1$, $N_\omega(K, \lambda K) = N(K, \lambda K)$.*

## Corollary 17

*For any $0 < \lambda < 1$ and $d \in \mathbb{N}$ we have $N_\omega(\ell, \lambda\ell) = N(\ell, \lambda\ell)$ and $N_\omega(C^d, \lambda C^d) = N(C^d, \lambda C^d)$ and so*

$$\mathrm{fcoin}(\ell) = \mathrm{coin}(\ell) = 4,$$

$$\mathrm{fcoin}(C^d) = \mathrm{coin}(C^d) = 2^{d+1}.$$

## Theorem 18

*Let* $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ *be a decomposition of* $\mathbb{E}^d$ *into the direct vector sum of its linear subspaces* $\mathbb{L}_i$ *and let* $K_i \subseteq \mathbb{L}_i$ *be convex bodies, then*

$$\mathrm{fcoin}(K_1 \oplus \cdots \oplus K_n) \leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^{n} N_\omega(K_i, \lambda K_i)}{1-\lambda} < \prod_{i=1}^{n} \mathrm{fcoin}(K_i). \qquad (3)$$

*and if* $K_i'$*s are o-symmetric,*

$$\overline{\mathrm{fcoin}}(K_1 \oplus \cdots \oplus K_n) = \inf_{\lambda = \frac{1}{2}} \frac{\prod_{i=1}^{n} \overline{N}_\omega(K_i, \lambda K_i)}{1-\lambda} < \prod_{i=1}^{n} \overline{\mathrm{fcoin}}(K_i). \qquad (4)$$

Relation (4), follows from

$$\overline{N}_\omega(K_1 \oplus K_2, \lambda(K_1 \oplus K_2)) = \overline{N}_\omega(K_1, \lambda K_1)\overline{N}_\omega(K_2, \lambda K_2).$$

A detailed treatment of weighted covering numbers and fractional covering indices appears in our paper:

K. Bezdek and M. A. Khan, On fractional coverings, (*forthcoming*).