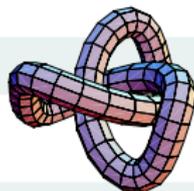


Cassini Sets in Generalized Minkowski Spaces

Thomas Jahn

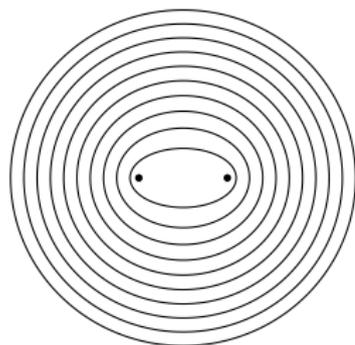
*Faculty of Mathematics
TU Chemnitz*



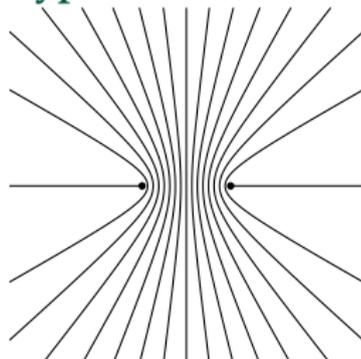
July 2, 2015

FUNCTIONS OF DISTANCES

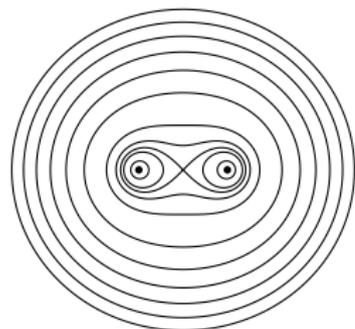
ellipses



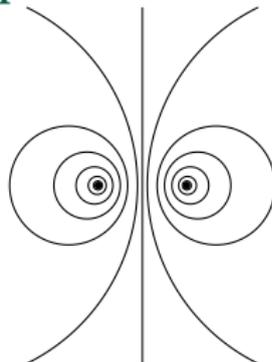
hyperbolas



Cassini curves



Apollonian circles



SETTING: GENERALIZED MINKOWSKI SPACES

A **norm** is a function

$$\|\cdot\| : \mathbb{R}^d \rightarrow \mathbb{R}$$

A **gauge** is a function

$$\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$$

such that

SETTING: GENERALIZED MINKOWSKI SPACES

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$$\gamma(x) \geq 0$$

nonnegativity

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location of zeros

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$$\gamma(\lambda x) = \lambda \gamma(x)$$

such that

homogeneity

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such that

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$$\|x + y\| \leq \|x\| + \|y\|$$

A **gauge** is a function

$$\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$$

$$\forall x, y \in \mathbb{R}^d, \lambda > 0$$

$$\gamma(x) \geq 0$$

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$$\gamma(\lambda x) = \lambda \gamma(x)$$

$$\gamma(x + y) \leq \gamma(x) + \gamma(y)$$

subadditivity = triangle inequality

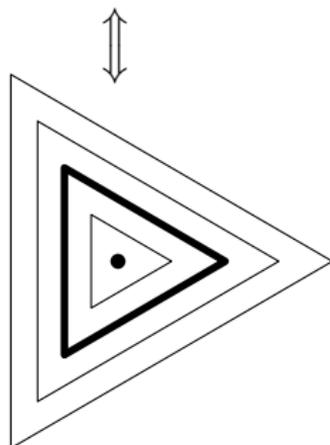
SETTING: GENERALIZED MINKOWSKI SPACES

Just like the special case of norms, gauges are uniquely determined by their **unit balls** $\{x \in \mathbb{R}^d \mid \gamma(x) \leq 1\}$.

SETTING: GENERALIZED MINKOWSKI SPACES

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$$\gamma(\xi_1, \xi_2) = \max \left\{ -\xi_1, \frac{\xi_1 + \sqrt{3}\xi_2}{2}, \frac{\xi_1 - \sqrt{3}\xi_2}{2} \right\}$$



SETTING: GENERALIZED MINKOWSKI SPACES

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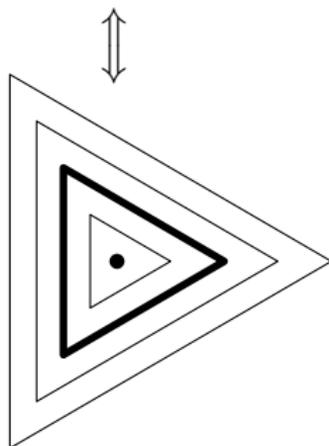
$$\gamma(\xi_1, \xi_2) = \max \left\{ -\xi_1, \frac{\xi_1 + \sqrt{3}\xi_2}{2}, \frac{\xi_1 - \sqrt{3}\xi_2}{2} \right\}$$

still compact

still convex

$0 \in \mathbb{R}^d$ still an interior point

not necessarily centered at $0 \in \mathbb{R}^d$



SETTING: GENERALIZED MINKOWSKI SPACES

Plenty of pleasant results from “classical functional analysis” in normed spaces $(\mathbb{R}^d, \|\cdot\|)$ also work in **generalized Minkowski spaces** (\mathbb{R}^d, γ) .



Cobzaş, Ş. (2013)

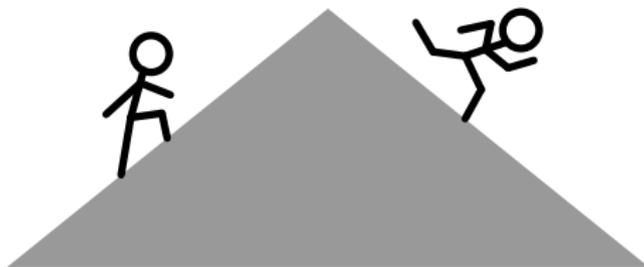
Functional Analysis in Asymmetric Normed Spaces,
Birkhäuser/Springer Basel AG, Basel.

<http://dx.doi.org/10.1007/978-3-0348-0478-3>

Most attention has to be paid when we interpret $\gamma(x - y)$ as a “distance”.

SETTING: GENERALIZED MINKOWSKI SPACES

Why should we consider asymmetric distance functions?



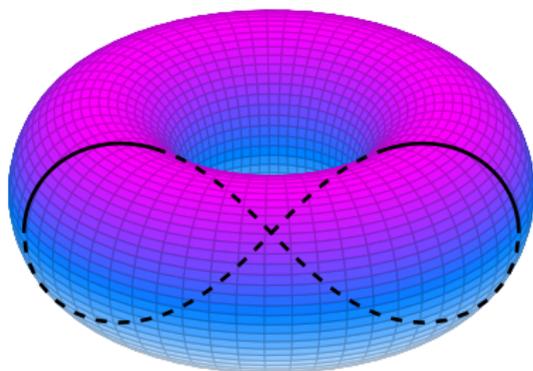
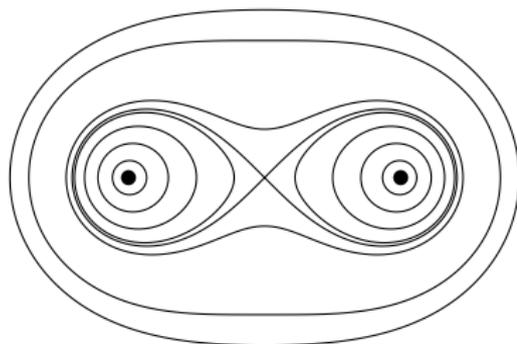
It makes sense in certain models in location science.

CASSINI CURVES: AN OVERVIEW

Given points $p_1, p_2 \in \mathbb{R}^2$, and a number $\alpha > 0$, what does the set

$$\{x \in \mathbb{R}^2 \mid \|x - p_1\|_2 \|x - p_2\|_2 = \alpha\}$$

look like?



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Given points $p_1, p_2 \in \mathbb{R}^2$, and a number $\alpha > 0$, what does the set

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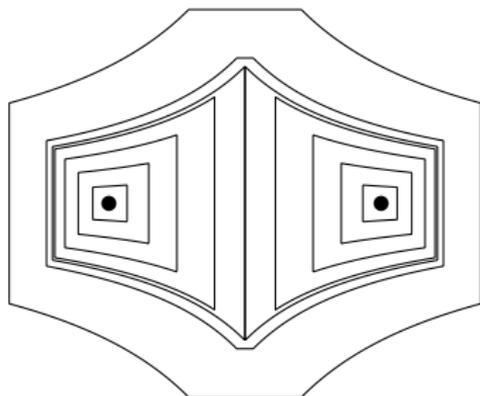
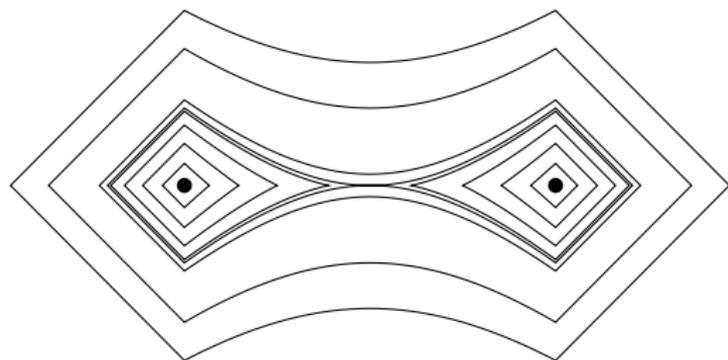


Martini, H., Wu, S. (2013).

Cassini curves in normed planes.

Results Math. 63(3–4):1159–1175.

CASSINI CURVES: AN OVERVIEW



CASSINI SETS: NEW CONTEXT

Given gauges $\gamma_1, \dots, \gamma_n$, points $p_1, \dots, p_n \in \mathbb{R}^d$, and a number $\alpha > 0$, what do the sets

$$\left\{ x \in \mathbb{R}^d \mid \gamma_1(x - p_1) \dots \gamma_n(x - p_n) = \alpha \right\},$$
$$\left\{ x \in \mathbb{R}^d \mid \gamma_1(x - p_1) \dots \gamma_n(x - p_n) \leq \alpha \right\}$$

look like?



J., Martini, H., Richter, C. (2016+).

Bi- and multifocal curves and surfaces for gauges.

J. Convex Anal., to appear.

STARSHAPEDNESS

Abbreviation: $f_{\leq \alpha} = \{x \in \mathbb{R}^d \mid f(x) \leq \alpha\}$

A coercivity lemma

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function such that the set of minimizers of f is non-empty and bounded. Then the following statements are true.

1. For every $\alpha \in \mathbb{R}$, the sublevel set $f_{\leq \alpha}$ is bounded.
2. There exists $\varrho \geq 0$ such that $f(\lambda_1 x) < f(\lambda_2 x)$ whenever $\|x\|_2 > \varrho$ and $1 \leq \lambda_1 < \lambda_2$.

STARSHAPEDNESS

Starshapedness of sublevel sets I

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be the pointwise product of finitely many convex and coercive functions. For α big enough, the sublevel set $f_{\leq \alpha}$ is star-shaped with respect to $x_0 = 0$.

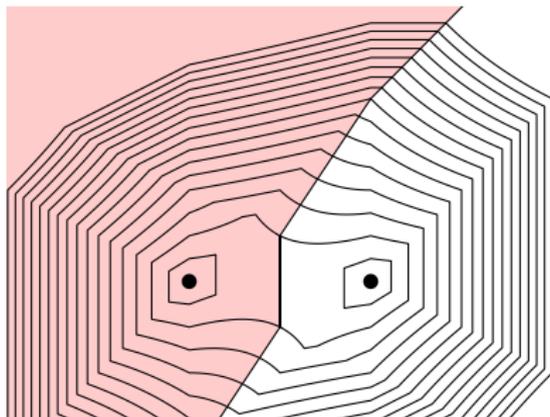
Starshapedness of sublevel sets II

Let $\gamma_1, \dots, \gamma_n : \mathbb{R}^d \rightarrow \mathbb{R}$ be gauges and let $p_1, \dots, p_n \in \mathbb{R}^d$ be arbitrary points. Define $f : \mathbb{R}^d \rightarrow \mathbb{R}$,
 $f(x) := \gamma_1(x - p_1) \cdot \dots \cdot \gamma_n(x - p_n)$, and let x_0 be fixed. For α big enough, the sublevel set $f_{\leq \alpha}$ is star-shaped with respect to x_0 .

DECOMPOSITION

Decomposition in normed spaces

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}, f(x) = \|x - p_1\| \|x - p_2\|$, and let V_1, V_2 be the Voronoi cells associated to $\{p_1, p_2\}$. Then, for each $\alpha \geq 0$ and $i \in \{1, 2\}$, the set $f_{\leq \alpha} \cap V_i$ is star-shaped with respect to p_i .



DECOMPOSITION

Decomposition in generalized Minkowski spaces

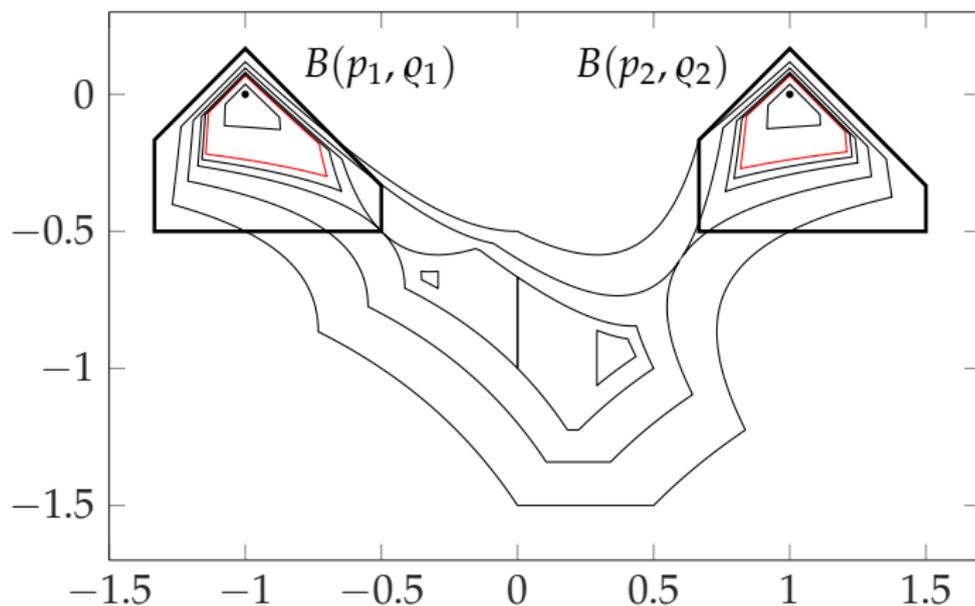
Let $f(x) = \gamma_1(x - p_1) \cdot \dots \cdot \gamma_n(x - p_n)$.

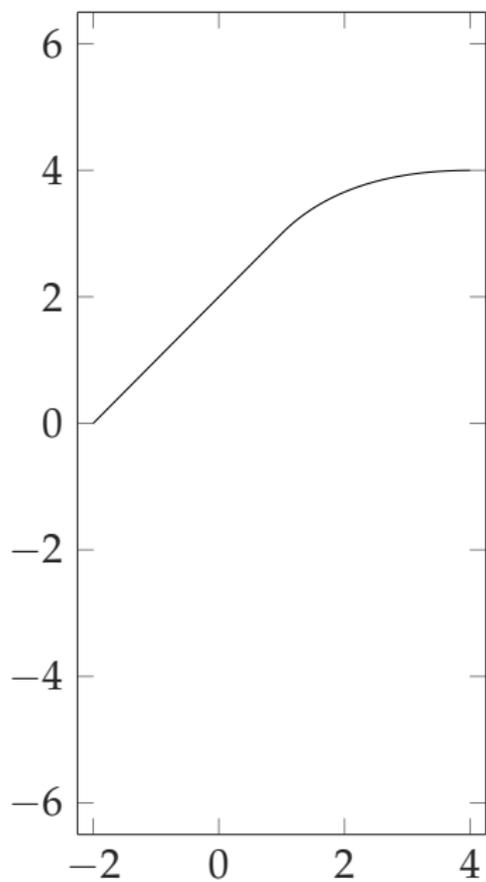
For q_i small enough and for all α ,

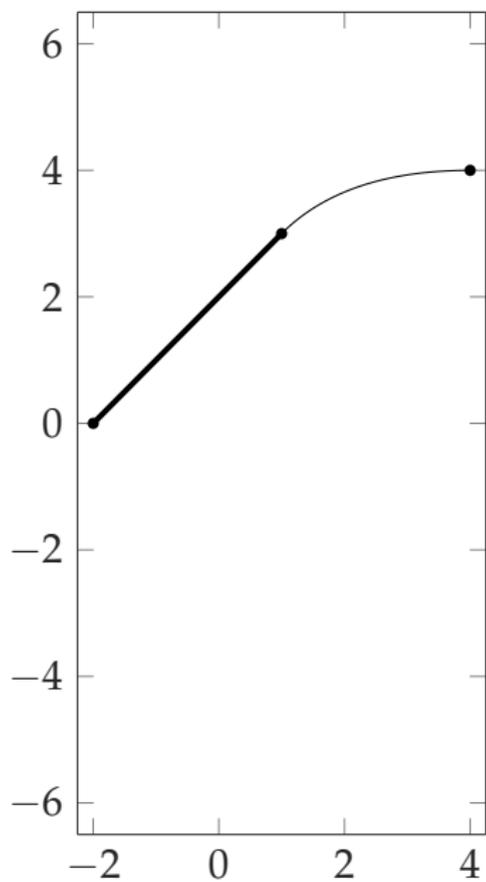
the set $f_{\leq \alpha} \cap (p_i + q_i B_{\gamma_i})$ is star-shaped with respect to p_i ,
and the interiors of the sets $p_i + q_i B_{\gamma_i}$ are mutually disjoint.

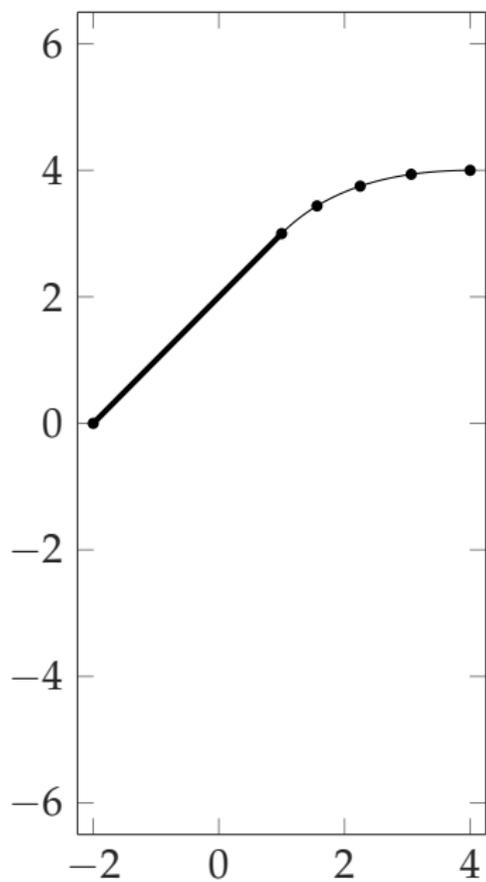
DECOMPOSITION

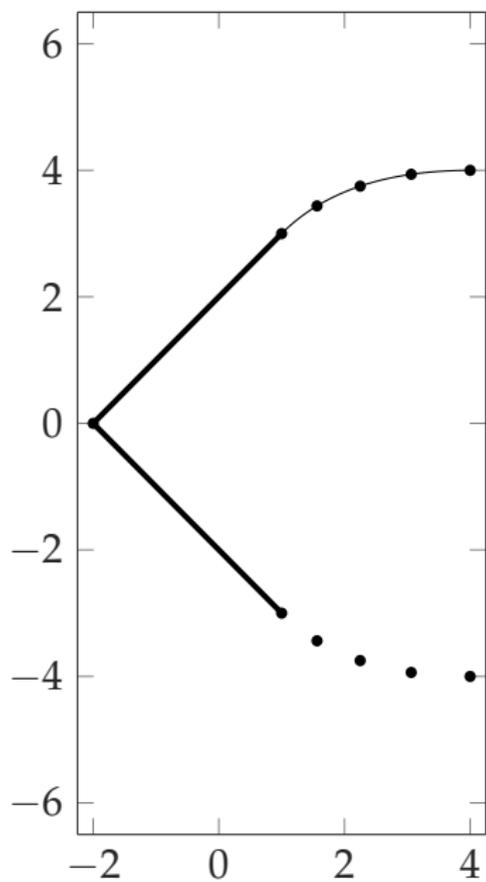
For α small enough, the sublevel set $f_{\leq \alpha}$ splits into exactly n connected components.

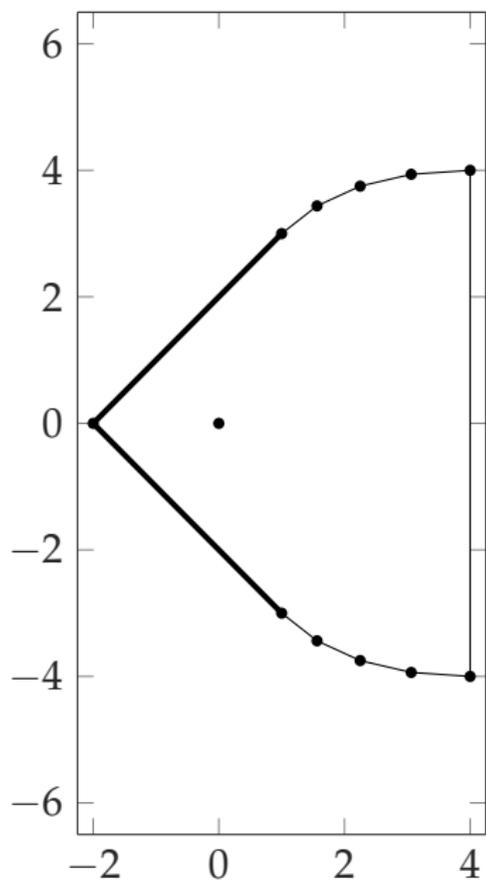


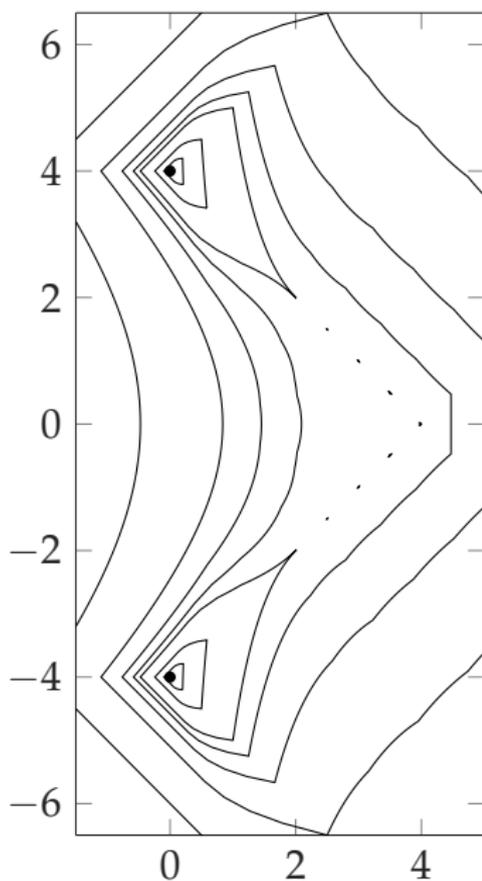
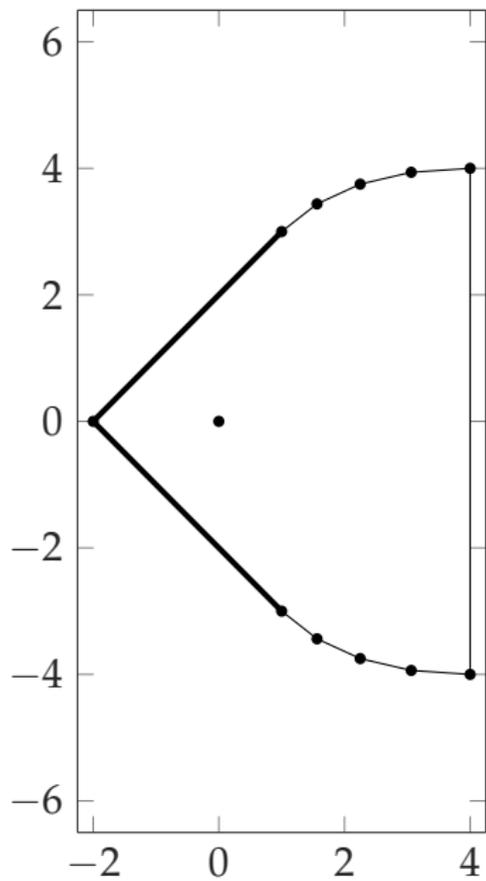












Thanks for your attention!