Colorings, monodromy, and impossible triangulations

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Geometry and Symmetry

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Impossible triangulations

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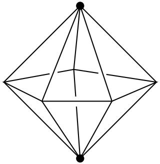
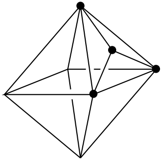
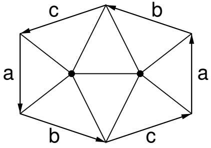
By contrast, one may have:

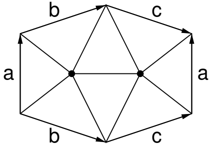
- Two non-adjacent or more than two adjacent odd vertices.





- Two adjacent on the torus and on the projective plane.

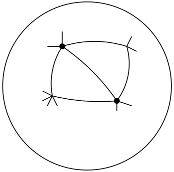




Reduction to even triangulations of polygons

First proof.

Assume such a triangulation exists.



Remove the edge joining the odd vertices (and the adjacent triangles). Get a square, triangulated with all vertices of even degree.

Thus, Theorem $\Leftrightarrow$ the square has no even triangulation. □

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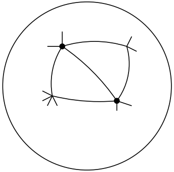

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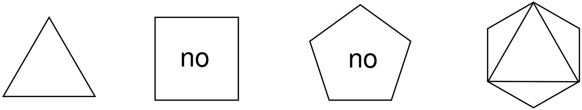
Lemma

An n-gon has a triangulation with all vertices of even degree ⇔ $n \equiv 0 (\mathrm{mod}\ 3)$.



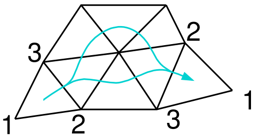
Even triangulations and colorings

Lemma

An n-gon has a triangulation with all vertices of even degree $\Leftrightarrow$ $n \equiv 0 \pmod 3$.

Proof.

An even triangulation can be vertex-colored in 3 colors:
color one triangle; this extends uniquely along any path;
extensions along different paths don't contradict,
due to the even degrees and to the simply-connectedness.



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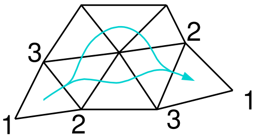

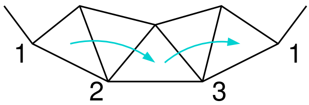

Even degrees $\Rightarrow$ colors of the boundary vertices repeat cyclically $1 \to 2 \to 3 \to 1 \to \cdots$. Hence n is divisible by 3. □

A generalization

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But the proof will look nicer in a different language...

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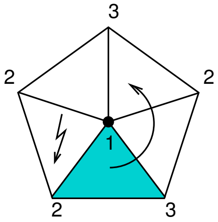

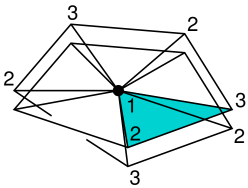

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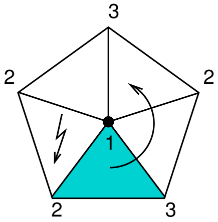

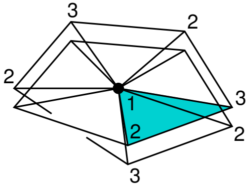

Compare: extending a holomorphic function $f\colon U \to \mathbb{C}$ along different paths can produce different values at the same point. These “branches” of f form the Riemann surface of f.

Coloring monodromy

Definition

Let M be a triangulated surface, Δ_0 a triangle in M, and $a_1, \ldots, a_n$ vertices of odd degree. The coloring monodromy

$$\pi_1(M \setminus \{a_1, \ldots, a_n\}, \Delta_0) \to \mathrm{Sym}(\Delta_0) \cong \mathrm{Sym}_3$$

is a group homomorphism that sends every path starting and ending at Δ_0 to the corresponding vertex re-coloring of Δ_0.

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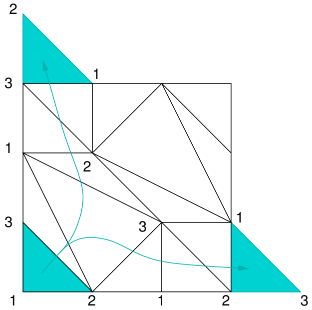
Example

In the 7-vertex triangulation of the torus all vertices have degree 6. The coloring monodromy

$$\mathbb{Z}^2 \cong \pi_1(M) \to \mathrm{Sym}_3$$

permutes the colors in a 3-cycle.



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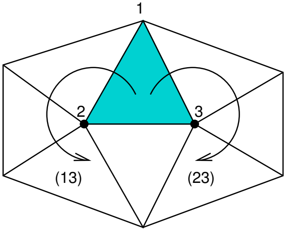
On the other hand, going around a and going around b permutes the colors by two different transpositions.



Hence the image must be the whole Sym_3. Contradiction.

References

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The associated branched cover was introduced and studied in

[I.-Joswig’03] Branched coverings, triangulations, and 3-manifolds.

(the focus was on triangulations of $\mathbb{S}^3$ with the edges of odd degrees forming a knot).

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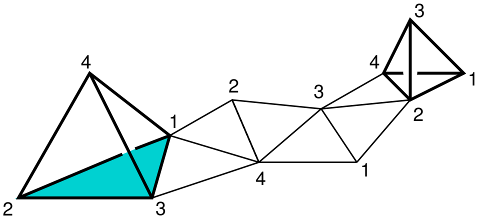
Match one of the faces of P with the base triangle of $\mathbb{S}^2$.

Rolling P along a closed path generates a symmetry of P.



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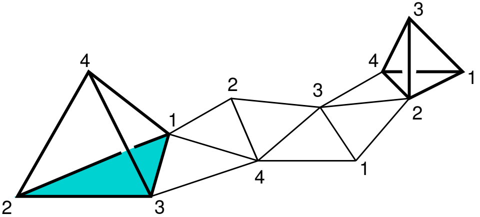

$$\mathbb{Z}^2 \cong \pi_1(\mathbb{S}^2 \setminus \{a, b\}) \to \mathrm{Sym}(P) \qquad (*)$$

Assume that a, b are adjacent and belong to the base triangle.

Then rolling around a and rolling around b produce two non-commuting symmetries of P.

Hence the image of $(*)$ is non-commutative. Contradiction.

The minimal colored cover

Definition

The minimal colored cover $\widetilde{\Sigma}$ *of a triangulated surface* Σ*:*

$$\{(\Delta, \varphi) \mid \Delta \in \Sigma,\ \varphi\colon \mathrm{Vert}(\Delta) \to \{1,2,3\}\} / \sim$$

Two adjacent colored triangles are glued along their common side if their colorings on that side agree.

This comes with a natural branched cover $\widetilde{\Sigma} \to \Sigma$.

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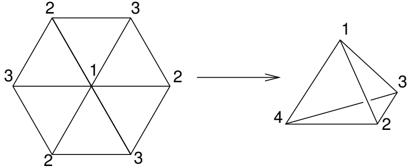

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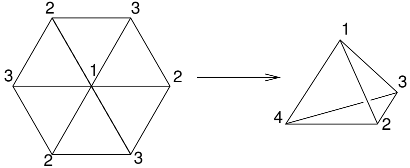
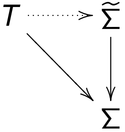


A universality property: if a colored surface covers Σ, then it also covers $\widetilde{\Sigma}$.



The space of germs

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Given two triangulated surfaces Σ, Σ'.
The space of germs $G(\Sigma, \Sigma')$ consists of triples

$$(\Delta, \Delta', \varphi), \quad \Delta \in \Sigma, \quad \Delta' \in \Sigma', \quad \varphi\colon \mathrm{Vert}(\Delta) \to \mathrm{Vert}(\Delta')$$

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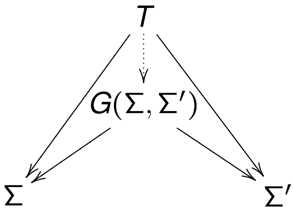
Naturally, $G(\Sigma, \Sigma')$ covers Σ and Σ'.

The universality property:
if a surface covers both Σ and Σ',
then it also covers $G(\Sigma, \Sigma')$.



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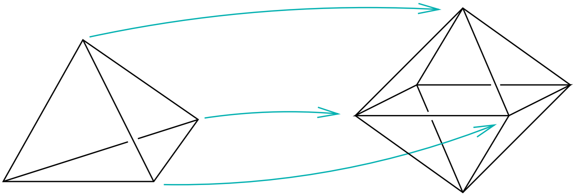

Example

Each connected component of $G(tetra, octa)$ is a regular map of the type $(3, 12)$ with $4 \cdot 8 \cdot 3 = 96$ faces. Hence it has 144 edges and 24 vertices, hence genus 13.
The group $\mathrm{Sym}(tetra) \times \mathrm{Sym}(octa)$ acts on $G(tetra, octa)$.

Spherical cone-metrics and the developing map

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Geometric proof.

Replace each triangle by an equiangular spherical triangle with the angle $\frac{2\pi}{k}$. Get a spherical metric on $\mathbb{S}^2$ with cone singularities: all except two are integer multiples of 2π.

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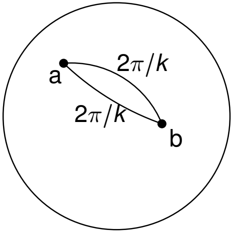
The complement of the edge between the exceptional vertices develops onto $\mathbb{S}^2$ with standard metric. The two sides of the slit go to two different geodesics of length $\frac{2\pi}{k}$ with the same endpoints. Contradiction.



□

Impossible torus triangulations and non-toral graphs

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As a corollary, every graph with these vertex degrees is not embeddable in the torus.