Approximations of round convex bodies in the plane and in higher dimensions

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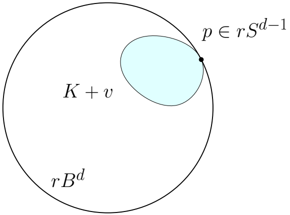
Happy birthday Károly and Egon!

Definition

A convex body $K \subset \mathbb{R}^d$ slides freely in rB^d if for each boundary point p of rB^d, there is a translate $K + v$ of K with $p \in K + v$ and $K + v \subset rB^d$.



For a convex body $K \subset \mathbb{R}^d$, the following are equivalent:

1. K slides freely in rB^d.
2. K is a Minkowski summand of rB^d.
3. K is an r-hyperconvex (r-spindle convex) body, that is, together with any two points $x, y \in K$, the intersection of all radius r closed balls containing x and y is also in K.

- The concept of hyperconvexity or spindle convexity (other names are also used in the literature) goes back a long way, for example to Mayer (1935), and possible even further.
- It seems that the idea appeared in various settings and guises in the literature so it is difficult to tell exactly when it started.
- Recent overviews of the topic can be found, for example, in the papers by Bezdek, Lángi, Naszódi and Papez (2007), and in Kupitz, Martini and Perles (2010).
- From among the many valuable contributions to this topic, I would like to single out the following paper which served us as a motivation

[1] Károly Bezdek, Zsolt Lángi, Márton Naszódi, and Peter Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), no. 2, 201–230.

- Let $x_1, \dots x_m \in \mathbb{R}^d$. The intersection of all radius r closed balls containing $x_1, \dots, x_m$ is denoted by $[x_1, \dots, x_m]_r$ (r-hyperconvex or r-spindle convex hull).
- **Note** that if K slides freely in rB^d and $x_1, \dots, x_m \in K$, then $[x_1, \dots, x_m]_r \subseteq K$.
- A ball-polytope or radius r is the r-hyperconvex hull of a finite set of points in $\mathbb{R}^d$.
- The intersection of a finite family of circular discs of radius r in $\mathbb{R}^2$ is called disc-polygon of radius r.
- A convex disc K that slides freely in a circle of radius r has a C^2_{++} smooth boundary if ∂K is C^2 and the curvature $\kappa(x) > 1/r$ for all $x \in \partial K$.
- The following asymptotic formulas are analogues of the corresponding results of L. Fejes Tóth (1953) and of McClure and Vitale (1975) about linearly convex discs.

Theorem (F.F., V. Vígh (2012) [4])

Let K be a convex disc that slides freely in a circle or radius r and has C^2_{++} smooth boundary. Then the following hold as $n \to \infty$,

$$\delta_\ell(K, K_n^\ell) \sim \frac{1}{24}\left(\int_{\partial K}\left(\kappa^2(s) - \frac{1}{r^2}\right)^{\frac{1}{3}} ds\right)^3 \cdot \frac{1}{n^2},$$

$$\delta_a(K, K_n^a) \sim \frac{1}{12}\left(\int_{\partial K}\left(\kappa(s) - \frac{1}{r}\right)^{\frac{1}{3}} ds\right)^3 \cdot \frac{1}{n^2},$$

$$\delta_H(K, K_n^H) \sim \frac{1}{8}\left(\int_{\partial K}\left(\kappa(s) - \frac{1}{r}\right)^{\frac{1}{2}} ds\right)^2 \cdot \frac{1}{n^2},$$

where K_n^ℓ, K_n^a, and K_n^H denote disc-polygons of radius r with at most n sides inscribed in K that are closest to K with respect to perimeter-deviation, area-deviation, and Hausdorff-metric, respectively.

We proved similar formulas for the circumscribed cases as well.

- Confirming a conjecure of Kershner, Dowker proved in 1944 that the maximum area of n-gons inscribed in a convex disc K is a concave function of n, and the minimum area of n-gons circumscribed about K is a convex function of n.
- L. Fejes Tóth (1955), Molnár (1955), and Eggleston (1957) observed independently of each other that Dowker's results are also true for perimeter.
- The theorems of Dowker and their extensions for the perimeter hold also on the sphere and in the hyperbolic plane.
- This was shown by Molnár in 1955 with the exception of the case of perimeter of circumscribed polygons on the sphere. This last case was settled by L. Fejes Tóth in 1958.
- The statements of the following theorem were proved by Bezdek, Lángi, Naszódi and Papez in 2007 for the special case when K is a closed circular disc of radius $r < 1$.

A Dowker-type result

- Let $a_i(n)$ and $p_i(n)$ denote the maximum area and maximum perimeter of convex disc-polygons (of radius r) with at most n vertices contained in K.
- Let $a_c(n)$ and $p_c(n)$ denote the minimum area and minimum perimeter of convex disc-polygons (of radius r) with at most n vertices containing K.

Theorem (G. Fejes Tóth, F.F. (2015) [2])

We have, for $n \geq 4$,

i) $a_c(n-1) + a_c(n+1) \geq 2a_c(n)$,

ii) $p_c(n-1) + p_c(n+1) \geq 2p_c(n)$,

iii) $a_i(n-1) + a_i(n+1) \leq 2a_i(n)$,

iv) $p_i(n-1) + p_i(n+1) \leq 2p_i(n)$.

We also proved that (i)–(iv) hold in the hyperbolic plane, and (i), (iii), (iv) hold on the sphere.

- Eggleston (1957) proved that the minimum area deviation and the minimum perimeter deviation of a convex n-gon from a convex disc K are concave functions of n.
- It is unknown whether the same holds for disc-n-gons of radius r and r-hyperconvex discs.
- To answer this question seems to be difficult. Here is one the reasons why.
- Eggleston (1957) proved that for a convex disc K among all convex n-gons the one closest to K in the sense of perimeter deviation is always inscribed in K.
- Direct computations show that if K is the circle of radius 0.9 and $n = 5$, then the best approximating disc-5-gon in the sense of perimeter deviation is neither inscribed in nor circumscribed about K.

A reverse isoperimetric inequality

K. Bezdek conjectured that among convex bodies of a given surface area that slide freely in a ball of radius r, the r-spindle is the unique body that has minimal volume.

Theorem (F.F.,Á. Kurusa, V. Vígh (2015) [1])

The r-spindle has minimal area among discs of equal perimeter that slide freely in a circle of radius r.

We note that the argument of the proof of the above theorem does not yield that the uniqueness of the r-spindle.

Conjecture (F.F.,Á. Kurusa, V. Vígh (2015) [1])

If the volume of a convex body K that slides freely in a ball of radius r is sufficiently close to that of an r-spindle K' of the same surface area, then K is close to K' in the Hausdorff metric of compact sets.

A Blaschke–Santaló type theorem

- Let the convex body $K \subset \mathbb{R}^d$ slide freely in a ball or radius r. The *r*-hyperconvex dual K^r of K is the collection of the centres of all closed balls of radius r that contain K.
- We define the *r*-hyperconvex volume product of K as

$$\mathcal{P}(K) := \mathrm{Vol}(K)\,\mathrm{Vol}(K^r).$$

Theorem (F.F.,Á. Kurusa, V. Vígh (2015) [1])

If $K \subset \mathbb{R}^d$ slides freely in a ball of radius r, then

$$\mathcal{P}(K) \leq \mathcal{P}\left(\frac{r}{2}B^d\right).$$

Equality holds if and only if $K = r/2 \cdot B^d + \mathbf{z}$ for some $\mathbf{z} \in \mathbb{R}^d$.

Stability of the volume product inequality

Theorem (F.F.,Á. Kurusa, V. Vígh (2015) [1])

Let $r > 0$, then there exist constants $c_{d,r} > 0$ and $\varepsilon_{d,r} \in (0, \frac{1}{2})$ depending only on d and r, and a monotonically decreasing positive real function $\mu(\varepsilon)$ with $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that a convex body $K \subset \mathbb{R}^d$ that slides freely in a ball of radius r satisfies

$$\mathcal{P}(K) \geq (1-\varepsilon)\mathcal{P}\left(\frac{r}{2}B^d\right)$$

for some $\varepsilon \in [0, \varepsilon_{d,r}]$ if and only if there exists a vector $z \in \mathbb{R}^d$ such that

$$\delta_H\left(K, \frac{r}{2}B^d + z\right) \leq c_{d,r}\mu(\varepsilon),$$

where $\delta_H(\cdot\,,\,\cdot)$ denotes the Hausdorff distance of compact sets.

Uniform random approximations

- Let $K \subset \mathbb{R}^d$ be a convex body that slides freely in rB^d.
- Let $x_1, \dots, x_n$ be a sample of i.i.d. random points from K chosen according to the uniform probability distribution.
- $K_n^r := [x_1, \dots, x_n]_r$ is a (uniform) random ball-polytope of radius r in K.
- The boundary ∂K is C^2_{++} smooth if it is C^2 and in any $x \in \partial K$ all principal curvatures are strictly larger than $1/r$.
- The following theorem implies, in the limit as $r \to \infty$, the corresponding asymptotic formulas of Rényi and Sulanke (1963, 1964) for convex discs K whose boundary is C^2_+ (C^5_+) smooth.

Theorem (F.F., P. Kevei, V. Vígh (2014) [3])

If the convex disc $K \subset \mathbb{R}^2$ has C^2_{++} boundary and slides freely in a circle of radius r, then

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^r)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{\frac{1}{3}} dx,$$

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n^r)) \cdot n^{\frac{2}{3}} = \sqrt[3]{\frac{2A(K)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{\frac{1}{3}} dx.$$

If, in additon, ∂K is C^5_{++} smooth, then

$$\lim_{n\to\infty} \mathbb{E}(P(K) - P(K_n^r)) \cdot n^{2/3}$$
$$= \frac{(12A(K))^{2/3}}{36} \Gamma\left(\frac{2}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{1/3} \left(3\kappa(x) + \frac{1}{r}\right) dx.$$

The case of the unit disc

Theorem (F.F., P. Kevei, V. Vígh (2014) [3])

If K is the unit radius closed circular disc, the following holds.

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^1)) = \frac{\pi^2}{2},$$

and

$$\lim_{n\to\infty} E(A(K \setminus K_n^1)) \cdot n = \frac{\pi^2}{2}.$$

[1] F. Fodor, Á. Kurusa, and V. Vígh, *Inequalities for hyperconvex sets*, Adv. Geom. (2015), to appear.

[2] G. Fejes Tóth and F. Fodor, *Dowker-type theorems for hyperconvex discs*, Period. Math. Hungar. **54** (2015), no. 1, 182–194.

[3] F. Fodor, P. Kevei, and V. Vígh, *On random disc-polygons in smooth convex discs*, Adv. in Appl. Probab. **46** (2014), no. 4, 899–918.

[4] F. Fodor and V. Vígh, *Disc-polygonal approximations of planar spindle convex sets*, Acta Sci. Math. (Szeged) **78** (2012), no. 1–2, 331–350.

Thank you for your attention.