On the Volume of Boolean Expressions of Large Congruent Balls

Balázs Csikós

Eötvös Loránd University & Alfréd Rényi Institute of Mathematics
Budapest

Geometry and Symmetry
29 June – 3 July 2015, Veszprém, Hungary
Dedicated to Károly Bezdek and Egon Schulte on the occasion of their 60th birthdays

Motivations

Kneser–Poulsen Conjecture (1954–55)

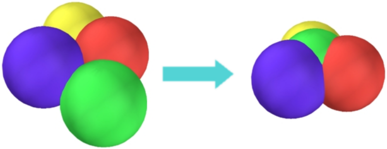
Suppose that for the points $\mathbf{p}_1, \ldots, \mathbf{p}_N$ *and* $\mathbf{q}_1, \ldots, \mathbf{q}_N$ *in* $\mathbb{R}^d$,

$$\|\mathbf{p}_i - \mathbf{p}_j\| \geq \|\mathbf{q}_i - \mathbf{q}_j\| \quad \forall\, 1 \leq i < j \leq N.$$

Do these inequalities imply for the inequality

$$\mathrm{Vol}_d\left(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r)\right) \geq \mathrm{Vol}_d\left(\bigcup_{i=1}^{N} B(\mathbf{q}_i, r)\right)$$

for $r > 0$*?*



Theorem (K. Bezdek, R. Connelly)

If there exist continuous curves $\gamma_i\colon [0,1] \to \mathbb{R}^{d+2}$ *such that* $\gamma_i(0) = \mathbf{p}_i$, $\gamma_i(1) = \mathbf{q}_i$, *and* $\|\gamma_i - \gamma_j\|\|$ *is a decreasing function for all* $1 \le i < j \le N$, *then*

$$\mathrm{Vol}_d\left(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r)\right) \ge \mathrm{Vol}_d\left(\bigcup_{i=1}^{N} B(\mathbf{q}_i, r)\right)$$

and

$$\mathrm{Vol}_d\left(\bigcap_{i=1}^{N} B(\mathbf{p}_i, r)\right) \le \mathrm{Vol}_d\left(\bigcap_{i=1}^{N} B(\mathbf{q}_i, r)\right)$$

for $r > 0$.

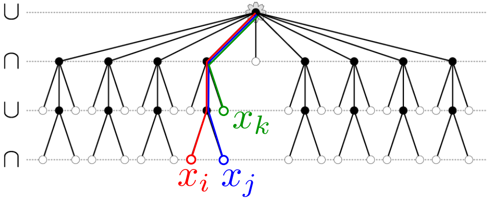
Leapfrog Lemma

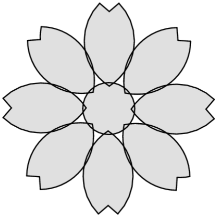
For any two systems of points $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ *in* $\mathbb{R}^d$, *there exist continuous curves* $\gamma_i\colon [0,1] \to \mathbb{R}^{2d}$ *such that* $\gamma_i(0) = \mathbf{p}_i$, $\gamma_i(1) = \mathbf{q}_i$, *and* $\|\gamma_i - \gamma_j\|\|$ *is a monotonous fuction for all* $1 \le i < j \le N$.

Corollary

The Kneser–Poulsen conjecture is true in $\mathbb{R}^2$.

- Let f be formal expression built from the variables $x_1, \ldots, x_N$ and the operation symbols $\cup$ and $\cap$ such that each variable appears in the expression exactly once.
- f can be visualized by a rooted tree in which leaves are labelled by the variables $x_1, \ldots, x_N$, inner nodes are labelled by $\cap$ and $\cup$ symbols.





- The sign $\epsilon_{ij} \in \{\pm 1\}$ is -1 if the paths from the vertices x_i and x_j first meet at a vertex labelled by $\cap$, $\epsilon_{ij} = 1$ otherwise.

Definition

Evaluation of f on some balls is a flower.

Theorem (B. Csikós)

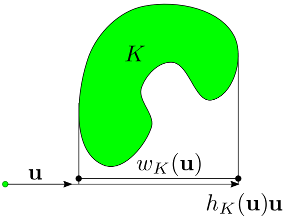
If there exist continuous curves $\gamma_i \colon [0,1] \to \mathbb{R}^{d+2}$ *such that* $\gamma_i(0) = \mathbf{p}_i$, $\gamma_i(1) = \mathbf{q}_i$, *and* $\epsilon_{ij}\|\gamma_i - \gamma_j\|$ *is a decreasing function for all* $1 \le i < j \le N$, *then*

$$\mathrm{Vol}_d\left(f(B(\mathbf{p}_1, r)\dots, B(\mathbf{p}_N, r))\right) \ge \mathrm{Vol}_d\left(f(B(\mathbf{q}_1, r)\dots, B(\mathbf{q}_N, r))\right).$$

Definition

For a bounded set $K \subset \mathbb{R}^d$ we define

- the support function $h_K \colon \mathbb{S}^{d-1} \to \mathbb{R}$,
  $h_K(\mathbf{u}) = \sup_{\mathbf{p}\in K}\langle \mathbf{u}, \mathbf{p}\rangle$;
- the width in the direction of $\mathbf{u} \in \mathbb{S}^{d-1}$,
  $w_K(\mathbf{u}) = h_K(\mathbf{u}) + h_K(-\mathbf{u})$;
- the mean width



$$w^d(K) = \frac{1}{d\omega_d}\int_{\mathbb{S}^{d-1}} w_K(\mathbf{u})\mathrm{d}\mathbf{u} = \frac{2}{d\omega_d}\int_{\mathbb{S}^{d-1}} h_K(\mathbf{u})\mathrm{d}\mathbf{u}.$$

Proposition

For a fixed system of centers $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \in (R^d)^N$, *if* r *is large, then*

$$\mathrm{Vol}_d\Big(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r)\Big) = \omega_d\Big(r^d + \frac{d}{2} w(\mathbf{p}) r^{d-1} + O(r^{d-2})\Big).$$

Theorem (I. Gorbovickis)

... and

$$\mathrm{Vol}_d\Big(\bigcap_{i=1}^{N} B(\mathbf{p}_i, r)\Big) = \omega_d\Big(r^d - \frac{d}{2} w(\mathbf{p}) r^{d-1} + O(r^{d-2})\Big).$$

Theorem (V. Sudakov, R. Alexander, V. Capoyleas, J. Pach)

If $\|\mathbf{p}_i - \mathbf{p}_j\| \geq \|\mathbf{q}_i - \mathbf{q}_j\|$ *for all* $i < j$, *then* $w(\mathbf{p}) \geq w(\mathbf{q})$.

Theorem (I. Gorbovickis)

Equality holds $\iff$ $\mathbf{p}$ *is congruent to* $\mathbf{q}$.

Boolean expressions

- $\mathcal{B}_N = \mathcal{B}[x_1, \ldots, x_N]$ – free Boolean algebra
- binary operations $\cup$, $\cap$, $\setminus$,
- complement operation $f \mapsto \bar{f}$,
- $\emptyset$ and X the minimal and maximal elements respectively.
- For $I \subseteq [N] = \{1, \ldots, N\}$ define the atomic expression a_I by

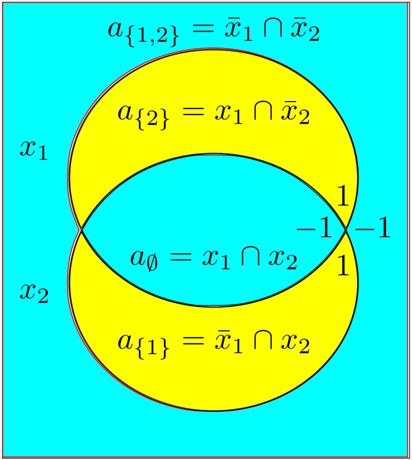
$$a_I = \Big(\bigcap_{j \notin I} x_j\Big) \cap \Big(\bigcap_{i \in I} \bar{x}_j\Big)$$

- Every Boolean expression $f \in \mathcal{B}_N$ has a unique decomposition

$$f = \bigcup_{a_I \subseteq f} a_I.$$

- $|\mathcal{B}_N| = 2^{2^N}$.
- Reduced Euler characteristic: $\tilde{\chi}_N(f) = \sum_{a_I \subseteq f} (-1)^{|I|+1}$.

Example



- $N = 2$
- 4 atomic expressions
- $\tilde{\chi}_2 \colon \mathcal{B}_2 \to [-2, 2] \cap \mathbb{Z}$
- $\tilde{\chi}_2(x_1) = \chi_2(x_2) = 0$
- $\tilde{\chi}_2(x_1 \cap x_2) = -1$
- $\tilde{\chi}_2(x_1 \cup x_2) = 1$
- $\tilde{\chi}_2(x_1 \triangle x_2) = 2$
- $\tilde{\chi}_2(\bar{x}_1 \triangle x_2) = -2$

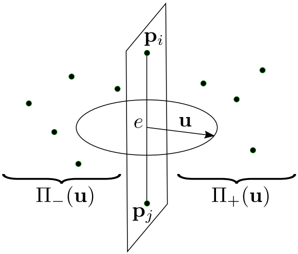
Boolean mean width

- $f \in \mathcal{B}_N$,
- $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$, points in general position,
- $e = [\mathbf{p}_i, \mathbf{p}_j]$ a segment connecting two points,
- $\mathbb{S}^{d-2}(e^\perp) = \{\mathbf{u} \mid \mathbf{u} \perp e$ and $\|\mathbf{u}\| = 1\}$,

We define a function

$$\nu_{f,e} \colon \mathbb{S}^{d-2}(e^\perp) \to \{-2, -1, 0, 1, 2\}$$

as follows.



$$y_k := \begin{cases} \emptyset & \text{if } \mathbf{p}_k \in \Pi_- \mathbf{u} \\ X & \text{if } \mathbf{p}_k \in \Pi_+ \mathbf{u} \\ x_k & \text{if } k \in \{i, j\} \end{cases}$$

$$f(y_1, \ldots, y_N) \in \mathcal{B}_2[x_i, x_j]$$

$$\nu_{f,e}(\mathbf{u}) := \tilde{\chi}_2(f(y_1, \ldots, y_N))$$

The Boolean mean width is defined as

$$w_f(\mathbf{p}) := \sum_e \left(\int_{\mathbb{S}^{d-2}(e^\perp)} \nu_{f,e}(\mathbf{u}) d\mathbf{u} \right) l_e.$$

Results

Theorem

For a fixed system of centers $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N) \in (\mathbb{R}^d)^N$, *if* $B_f(\mathbf{p}, r) = f(B(\mathbf{p}_1, r), \ldots, B(\mathbf{p}_N, r))$ *is bounded and* r *is large, then*

$$\mathrm{Vol}_d\big(B_f(\mathbf{p}, r)\big) = \omega_d\big(v_f r^d + \frac{d}{2} w_f(\mathbf{p}) r^{d-1} + O(r^{d-2})\big),$$

where $v_f = 1$ *if* $f \supseteq \bigcap_i x_i$, *and* $v_f = 0$ *otherwise.*

Theorem

If $f^*(x_1, \ldots, x_N) = \overline{f(\bar{x}_1, \ldots, \bar{x}_N)}$ *is the dual Boolean expression, then*

$$w_f = -w_{f^*}.$$

Theorem

If f *is in the free lattice generated by* $x_1, \ldots, x_N$, *then* f *can be evaluated on real numbers by* $a \cup b = \max\{a, b\}$ *and* $a \cap b = \min\{a, b\}$. *In that case,*

$$w_f(\mathbf{p}) = 2 \int_{\mathbb{S}^{d-1}} f(\langle \mathbf{p}_1, \mathbf{u} \rangle, \ldots, \langle \mathbf{p}_N, \mathbf{u} \rangle) \mathrm{d}\mathbf{u}.$$

Results

Theorem

Let f be an expression built from the variables $x_1, \ldots, x_N$ and the operations $\cup$ and $\cap$ so that each variable appears in the expression exactly once. Let ϵ_{ij} be the signs as earlier. Then if If $\epsilon_{ij}\|\mathbf{p}_i - \mathbf{p}_j\| \geq \epsilon_{ij}\|\mathbf{q}_i - \mathbf{q}_j\|$ for all $i < j$, then

$$w_f(\mathbf{p}) \geq w_f(\mathbf{q}).$$

Question

Can we have equality if $\mathbf{p}$ is not congruent to $\mathbf{q}$?

Corollary (IF YES)

If the answer to the question is yes, then under the conditions of the theorem above,

$$\mathrm{Vol}_d(B_f(\mathbf{p}, r)) \geq \mathrm{Vol}_d(B_f(\mathbf{p}, r))$$

for sufficiently large r, and a similar inequality holds also for the surface volume of the boundaries of these flowers.


HAPPY
BIRTHDAY