Bipartite Graphs and Distance Geometry

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Basic Problem: Molecule Problem

Given a graph G and a distance $d_{ij} = d_{ji} > 0$ associated with each edge of G, does there exist a configuration $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$, where each $\mathbf{p}_i \in \mathbb{R}^d$ is such that $|\mathbf{p}_i - \mathbf{p}_j| = d_{ij}$?

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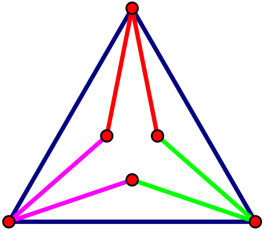
This problem is probably NP hard in general.

Impossibilities

For a given set of lengths, there may be no solution. For example, the triangle inequality must hold

$$|\mathbf{p}_i - \mathbf{p}_j| + |\mathbf{p}_j - \mathbf{p}_k| \geq |\mathbf{p}_i - \mathbf{p}_k|.$$

But similar inequalities must hold beyond the triangle inequality. For example, for four points the following set of distances do not exist in any dimension. (Like colored edges are identified.)



The Complete Graph

If ALL the pairwise distances between the vertices are given, finding the configuration, if it exists, is easy. Call $\mathbf{p}_1 = 0$, then the rank and positive semi-definiteness of the Gram matrix GM gives the answer by factoring:

$$GM = [\mathbf{p}_2, \dots, \mathbf{p}_n]^t [\mathbf{p}_2, \dots, \mathbf{p}_n] = [\mathbf{p}_i \cdot \mathbf{p}_j] = [d_{1i}^2 + d_{1j}^2 - d_{ij}^2]/2$$

Possibilities

If you are not fussy about the dimension of the affine span of your solution to the molecule problem, there is a reasonable solution.

SDP

Semi-definite programing

This is an effective algorithm, given the graph G and the distance set, that will either return a configuration *approximately* satisfying the distance constraints OR say there is no such configuration.

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But the configuration may exist in some humungus dimension.

Back to the Molecule Problem

When does SDP output a configuration in the desired dimension?

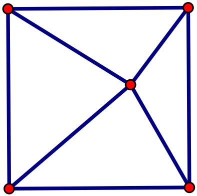
A word from our sponsor

UNIVERSAL RIGIDITY

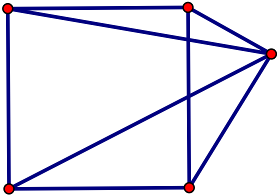
Universal rigidity

Given a graph G and a corresponding configuration $\mathbf{p}$ in $\mathbb{R}^d$, we say that $(G, \mathbf{p})$ is *universally rigid* if for any other configuration $\mathbf{q}$ in any $\mathbb{R}^D \supset \mathbb{R}^d$, with corresponding edge lengths in G the same, then $\mathbf{q}$ is congruent to $\mathbf{p}$. That is, ALL edge lengths of $\mathbf{q}$ are the same as the corresponding edge lengths of $\mathbf{p}$.

Planar examples



Not universally rigid



Universally rigid

Back to the molecule problem

A direct consequence of the definition of universal rigidity is:

Theorem

If $(G, \mathbf{p})$ is universally rigid, then the molecule problem, given its edge lengths, is approximately solvable by SDP.

The determination of universal rigidity is a much more tractable problem on its own.

Theorem (Connelly-Gortler 2014)

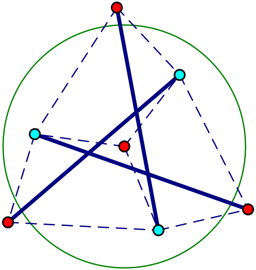
Given universally rigid $(G, \mathbf{p})$, in general, it is possible to find a certificate that guarantees that it is universally rigid.

The certificate above is a sequence of positive semi-definite matrices, whose ranks sum to $n - d - 1$ and another calculation that rules out affine motions.

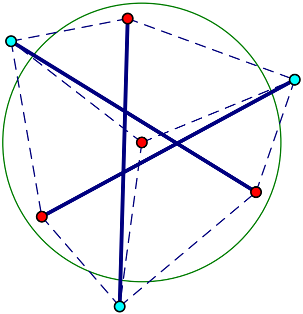
An interesting example: Complete bipartite graphs

Theorem (Connelly, Gortler 2015)

If $(K(n,m),(\mathbf{p},\mathbf{q}))$ *is a complete bipartite framework in* $\mathbb{R}^d$, *with* $m+n \geq d+2$, *such that the partition vertices* $(\mathbf{p},\mathbf{q})$ *are strictly separated by a quadric, then it is not universally rigid.*



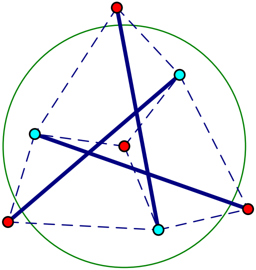
Universally Rigid



Not Universally Rigid

Symmetry Helps

If the configuration is symmetric about a point with a sufficiently large symmetry group the conic (or quadric in 3D) must also be symmetric by averaging, and this simplifies the calculation of universal rigidity considerably.



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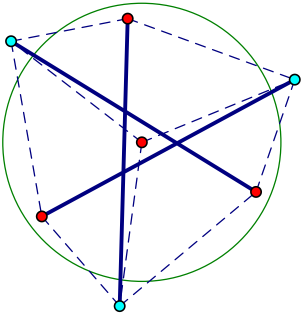


Not Universally Rigid

The Verenose Map

- Let $\mathcal{M}_d$ be the $(d+1)(d+2)/2$ dimensional space of $(d+1)$-by-$(d+1)$ symmetric matrices, which we call the *matrix space*.
- Define the map $\mathcal{V} : \mathbb{R}^d \to \mathcal{M}_d$ by $\mathcal{V}(\mathbf{v}) = \hat{\mathbf{v}}\hat{\mathbf{v}}^t$, which is a $(d+1)$-by-$(d+1)$ symmetric matrix, with the lower right-hand coordinate 1, where $\hat{\mathbf{v}}$ is the vector with an extra coordinate of 1 added at the bottom.
- $\mathcal{V}(\mathbb{R}^d)$ is a d dimensional algebraic set embedded in a $(d+1)(d+2)/2-1$ dimensional linear subspace. The function $\mathcal{V}$ is called the *Veronese map*.

The Verenose Map

The effect of $\mathcal{V}$ is to transfer quadratic conditions in $\mathbb{R}^d$ to linear conditions in a $(d+1)(d+2)/2-1$ dimensional linear subspace of $\mathcal{M}_d$.

Proposition

In $\mathbb{R}^d$ the vertices of the configurations $\mathbf{p}$ and $\mathbf{q}$ can be strictly separated by a quadric , if and only if the matrix configurations $\mathcal{V}(\mathbf{p})$ and $\mathcal{V}(\mathbf{q})$ can be strictly separated by a hyperplane in $\mathcal{M}_d$.

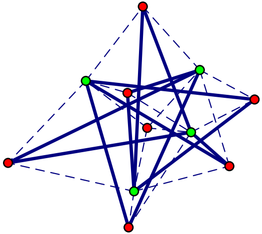
So in the plane the vertices of the partitions of a bipartite graph $K(4,3)$ can be strictly separated by a conic if and only if their Veronese images can be linearly separated by a hyperplane in a 5 dimensional linear space.

Non-separability Predicts Universal Rigidity

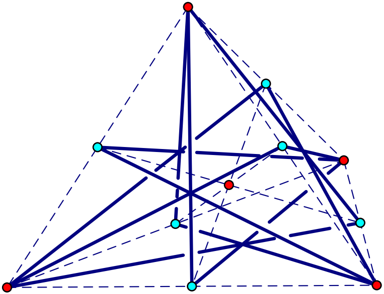
Theorem (Connelly, Gortler 2015)

If the convex hull of $\mathcal{V}(\mathbf{p})$ and $\mathcal{V}(\mathbf{q})$ intersect in the relative interior of each set, then $(K(m,n),(\mathbf{p},\mathbf{q}))$ is universally rigid.

Some examples in $\mathbb{R}^3$:



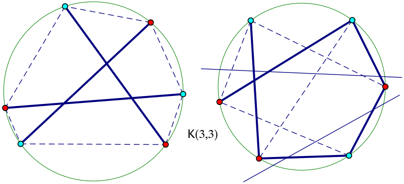
K(7,4)



K(6,5)

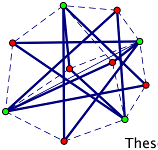
Lower Dimensional Spans

When the Veronese image of the configuration lies on a conic (or quadric) the separation criteria determine the universal rigidity.



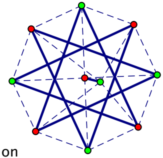
Universally Rigid

Not Universally Rigid

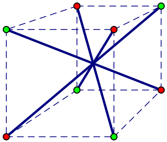


K(6,4)

These lie on one quadric.



K(5,5)



K(4,4)

This lies on the intersection of 3 quadrics.

Back to the Molecule Problem

- The generic case: This is where there is no algebraic relation among the coordinates of the configuration. Here the conic/quadric separation criterion determines whether its configuration can be determined with SDP.
- The generic realization of $K(5,5)$ in $\mathbb{R}^3$ has another realization with the same edge lengths. So SDP fails to give a 3 dimensional realization, or if it does, it may not be the "right" one. But when its vertices lie on a quadric and cannot be separated with a quadric, it can be universally rigid.
- Instead of general position or generic position, it is useful to assume that the Veronese realization is in general position. So SDP success is determined by the separability.

Acknowledgements

A paper "On universally rigid frameworks on the line" by Tibor Jordán and Viet-Hang Nguyen was a big impetus for our results here. They essentially gave the separation criterion for the case when the bipartite graph is in the line.

An old paper "When is a bipartite graph a rigid framework" by Ethan Bolker and Ben Roth was very insightful and laid the groundwork for this paper.