Chirality in discrete structures

Marston Conder
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Geometry and Symmetry
(Veszprém Hungary, June/July 2015)

Egon Schulte

Egon Schulte

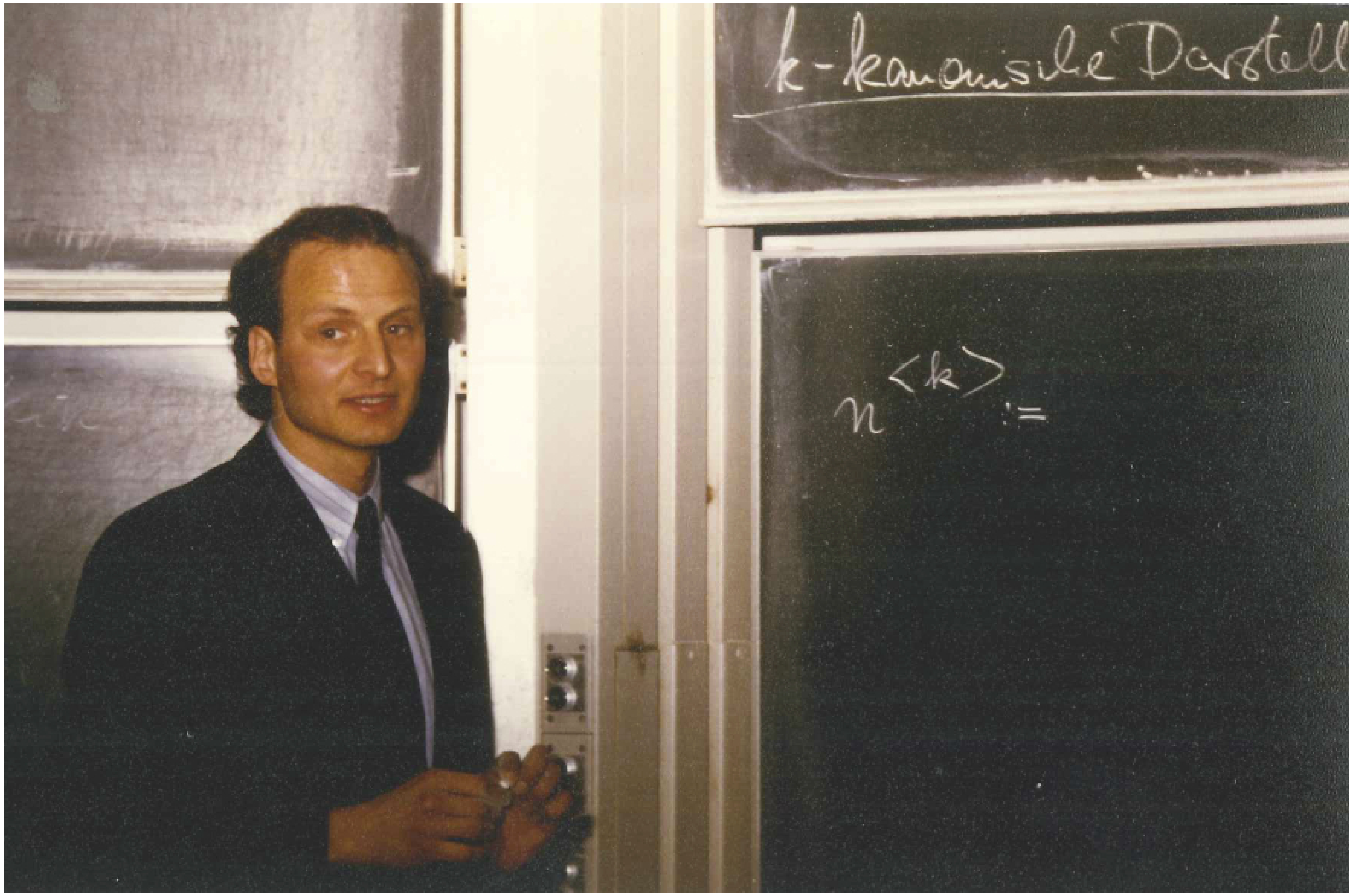
- Born 7 January 1955 in Heggen, Germany
- PhD in 1980, University of Dortmund, Germany
- Doctoral dissertation on Regular Incidence Complexes (aka Abstract Regular Polytopes)
  Supervisor: Ludwig Danzer
- Expert in discrete geometry (esp. polyhedra/polytopes), combinatorics & group theory
- Professor at Northeastern University, Boston

Egon Schulte - Publications

Abstract Regular Polytopes, book (2002) co-authored with Peter McMullen, Cambridge University Press

Approx. 90 other publications from 1981 onwards, many in top journals such as *Advances in Mathematics*, *Advances in Applied Mathematics*, *Advances in Geometry*, *Discrete and Computational Geometry*, *Geometriae Dedicata*, *Journal of Combinatorial Theory*, *Journal of the London Math Society*, *Journal für die Reine und Angewandte Mathematik*, *Proceedings of the London Math Society*, *Transactions of the American Math Society*, and the rapidly up-and-coming new journal *Ars Mathematica Contemporanea*

Co-authors: Peter McMullen, Asia Weiss, Jörg Wills, Barry Monson, Isabel Hubard, and 20 others ..



Egon Schulte - Habilitation (1985)

Peter McMullen, Asia Weiss and Egon - at Siófok (1985)

Egon in 1987

Working very hard in New Zealand (2010)

Near Asia's cottage in Ontario (2011)

With Asia at Ixtapa (2014)

Congratulations on your 60th birthday Egon!

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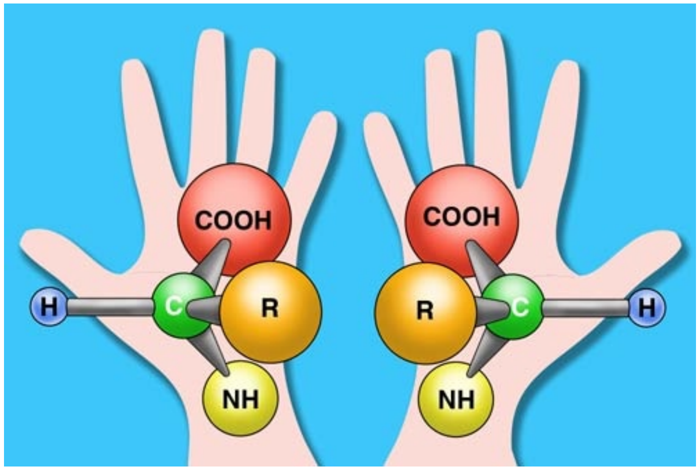
What is Chirality?

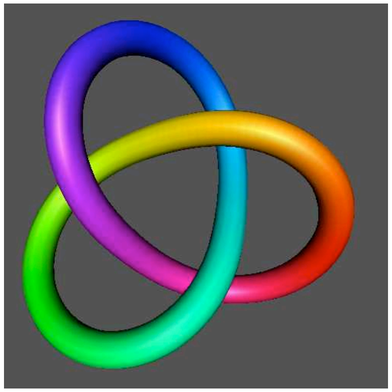
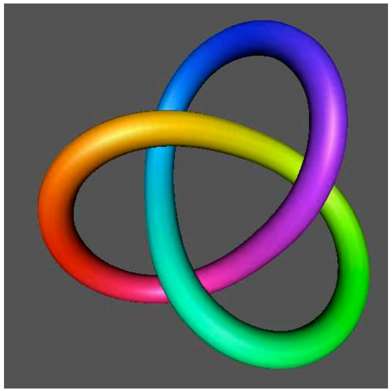
An object is called chiral if it differs from its mirror images.

History/terminology



The term 'chiral' means handedness, derived from the Greek word $\chi\epsilon\iota\rho$ (or ‘kheir’) for ‘hand’. It is usually attributed to the scientist William Thomson (Lord Kelvin) in 1884, although the philosopher Kant had earlier observed that left and right hands are inequivalent except under mirror image.

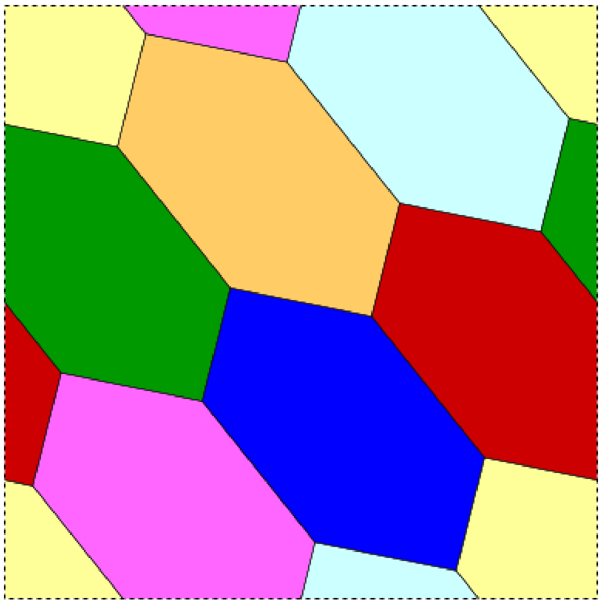
Chirality in mathematics

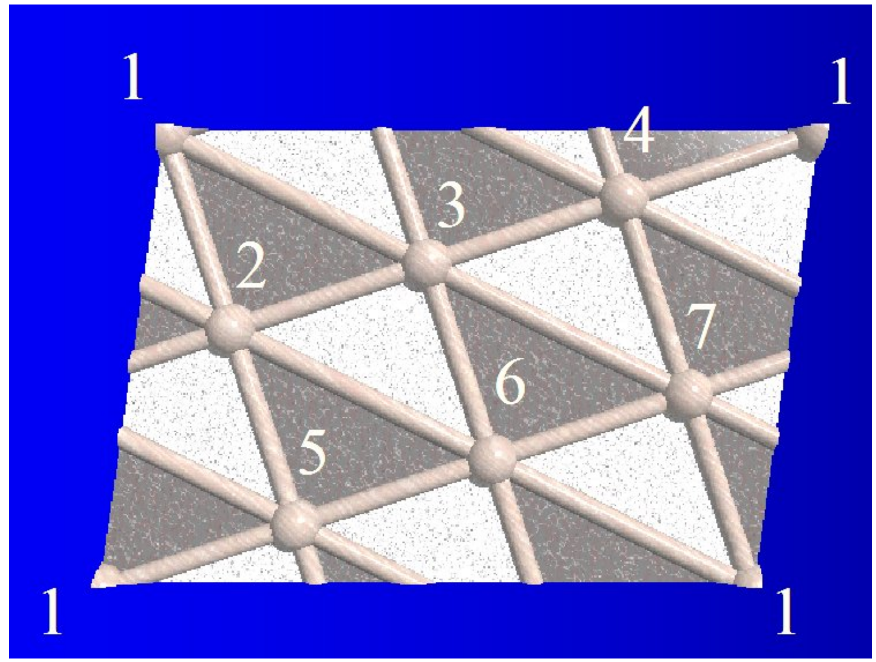
 

The right and left trefoil knots are inequivalent ... with Jones polynomials $t + t^3 - t^4$ and $t^{-1} + t^{-3} - t^{-4}$ respectively

Many of the other invariants of these knots (including their Alexander polynomials) are exactly the same for both, some because they are mirror images of each other, and in purely mathematical terms they have equal importance, but ...

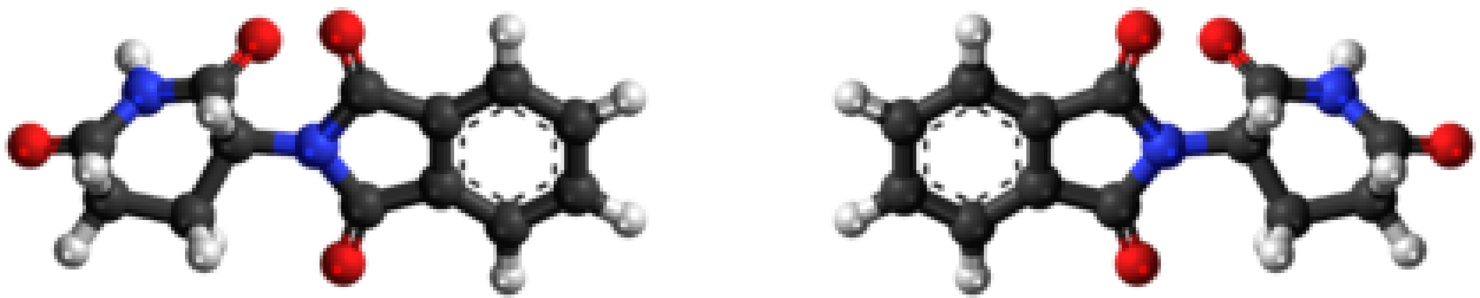
Maps of type $\{6,3\}$ and $\{3,6\}$ on the torus





These regular maps are chiral, and each is isomorphic to the dual of the other. (The one on the right is a triangulation of the torus using the complete graph K_7.)

Chirality in biology/chemistry/medicine



The two enantiomorphs of thalidomide have vastly different effects ... one is a sedative, but its mirror image causes birth defects ... making the context important

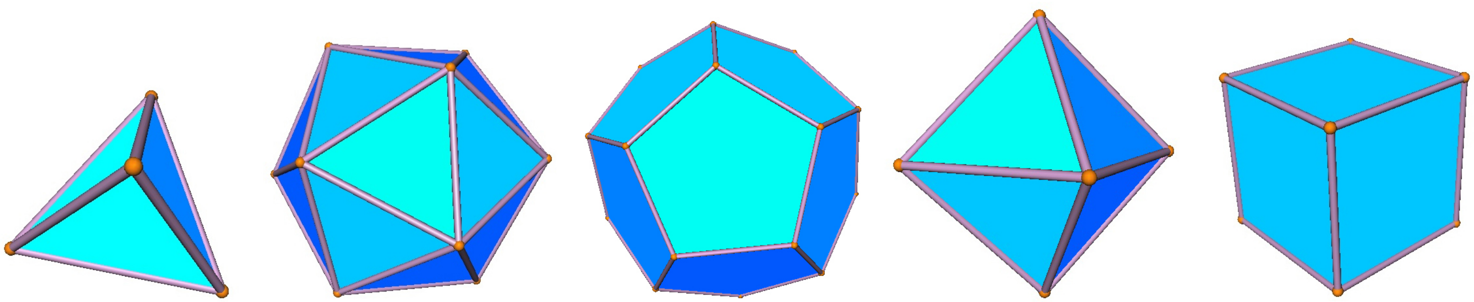
Similarly, differences between aspartame (sweetener) and its mirror image (bitter), and (S)-carvone (like caraway) and its mirror image (R)-carvone (like spearmint).

Chiral or reflexible?

In biological/chemical/medical/physical contexts we have no reason to expect mirror symmetry — so objects tend to be chiral — but the following is a remarkable phenomenon:

When a discrete object has a large degree of **rotational** symmetry, it often happens that it has also **reflectional** symmetry, so that chirality is not necessarily the norm

e.g. the Platonic solids are all reflexible!



Tetrahedron Icosahedron Dodecahedron Octahedron Cube

Open question: How prevalent is chirailty?

- for Riemann surfaces?
- for regular maps?
- for abstract polytopes?
- for other orientable discrete structures like these?

Riemann surfaces

A Riemann surface is a 1-dimensional complex manifold. More roughly speaking, a Riemann surface is an orientable surface endowed with some analytic structure.

An automorphism of a Riemann surface X is a structure-preserving homeomorphism from X to X, and this can be conformal or anticonformal, depending on whether it preserves or reverses the orientation of X.

Theorem [Hurwitz (1893)] A compact Riemann surface of genus $g > 1$ has at most $84(g-1)$ conformal automorphisms, and this upper bound is attained if and only if the conformal automorphism group $\mathrm{Aut}^+(X)$ is a (smooth) quotient of the ordinary $(2,3,7)$ triangle group $\langle\, x, y \,|\, x^2 = y^3 = (xy)^7 = 1 \,\rangle$.

Hurwitz surfaces of 'small' genus

Genus	Rfl	Ch
3	1	0
7	1	0
14	3	0
17	0	2
118	1	0
129	1	2
146	3	0
385	1	0
411	3	0
474	3	0
687	1	0

Genus	Rfl	Ch
769	3	0
1009	1	0
1025	0	8
1459	1	0
1537	1	0
2091	1	6
2131	3	0
2185	3	0
2663	0	2
3404	3	0
4369	3	0

Genus	Rfl	Ch
4375	1	0
5433	3	0
5489	0	2
6553	3	0
7201	1	4
8065	0	2
8193	1	12
8589	3	0
11626	1	0
11665	0	2
Total	50	42

Rfl = Reflexible Ch = Chiral

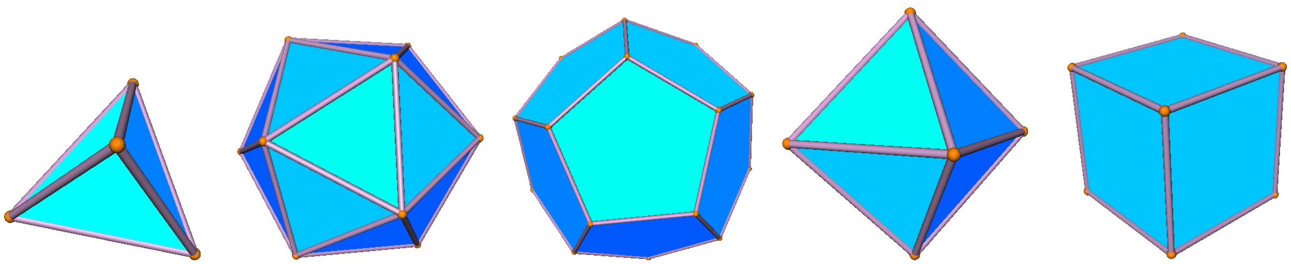
Orientably-regular maps

A map is an embedding of a connected graph or multigraph on a closed surface, breaking it up into simply-connected regions called the faces of the map.

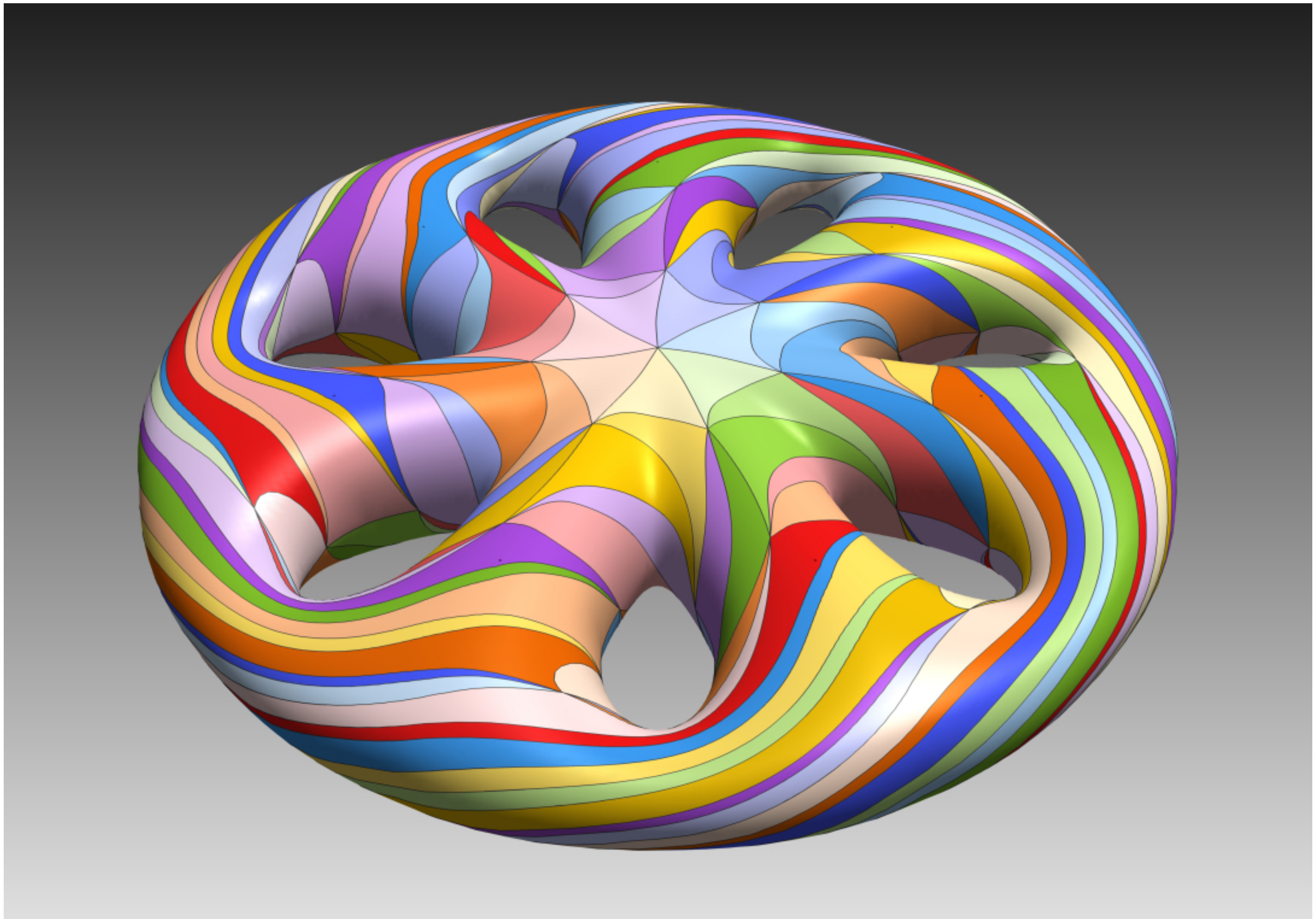
A map M on an orientable surface is orientably-regular if the group of all of its orientation-preserving automorphisms is transitive on the arcs (incident vertex-edge pairs) of M. In that case, every vertex has the same degree/valency m, and every face of the map has the same size k, and we call $\{k, m\}$ the type of the map.

Orientably-regular maps are sometimes just called 'regular'. Those that admit orientation-preserving automorphisms are 'reflexible', while the others are 'chiral'.

The Platonic solids give rise to reflexible maps on the sphere — with types $\{3,3\}$, $\{3,5\}$, $\{5,3\}$, $\{3,4\}$ and $\{4,3\}$:



Regular maps on the torus (genus 1) have types $\{3,6\}$, $\{4,4\}$ and $\{6,3\}$, and infinitely many of these are reflexible, and infinitely many are chiral.



A reflexible map of type $\{3,7\}$ on a surface of genus 7

Chirality among regular maps of small genus?

Rotary orientable maps of small genus:

Genus 2: 6 reflexible, 0 chiral
Genus 3: 12 reflexible, 0 chiral
Genus 4: 12 reflexible, 0 chiral
Genus 5: 16 reflexible, 0 chiral
Genus 6: 13 reflexible, 0 chiral
Genus 7: 12 reflexible, 4 chiral

Genus 2 to 100: 5972 reflexible, 1916 chiral (24% chiral)
Genus 101 to 200: 9847 reflexible, 4438 chiral (31%)
Genus 201 to 300: 10600 reflexible, 5556 chiral (34%)

Important open question: What about for larger genera?

Chiral maps/polyhedra of given type

By an amazing piece of work of Murray Macbeath (1969), it is known that for every hyperbolic pair (k,m) of positive integers (with $1/k + 1/m < 1/2$), there exist infinitely many orientably-regular maps of type $\{k,m\}$ (with rotation groups $\mathrm{PSL}(2,p)$ for various primes p). All of these maps are reflexible, and hence fully regular.

Question [Singerman (1992)]: What about chiral maps?

Theorem [Bujalance, MC & Costa (2010)]: For every $\ell \geq 7$, all but finitely many A_n are the automorphism group of an orientably-regular but chiral map of type $\{3,\ell\}$.

New Theorem (2014):

For every hyperbolic pair (k, m), there exist infinitely many orientably-regular but chiral maps of type $\{k, m\}$.

One 'base' example for each type can be found by

• constructing permutation representations of the ordinary $(2, k, m)$ triangle group [MC, Hucíková, Nedela & Širáň],

or by

• using group representations and the theory of differentials on Riemann surfaces [Jones].

Then infinitely many more examples of each type $\{k, m\}$ can be constructed by the 'Macbeath trick' for abelian p-covers.

Other consequences

1) **Chiral maps with simple underlying graphs**:

The automorphism groups of the base examples of both kinds are 'almost-simple', and in particular, have no cyclic normal subgroups. It follows that the vertex- and face- stabilisers are core-free in the automorphism group of the map, and hence for every hyperbolic pair (k, m), there exist at least two orientably-regular maps of type $\{k, m\}$, one reflexible and one chiral, such that both the map and its dual have simple underlying graph.

In fact there are infinitely many of each kind, and the same is known for the toroidal case (with $1/k + 1/m = 1/2$).

2) **Chiral polyhedra of every hyperbolic type**:

Also in each of these maps, every edge has two vertices and every edge lies in two faces, and therefore the maps are abstract polyhedra. Thus we have the following as well:

For every pair (k, m) of integers with $1/k + 1/m \leq 1/2$, there exist infinitely many regular and infinitely many orientably-regular but chiral polyhedra of type $\{k, m\}$.

What about polytopes?

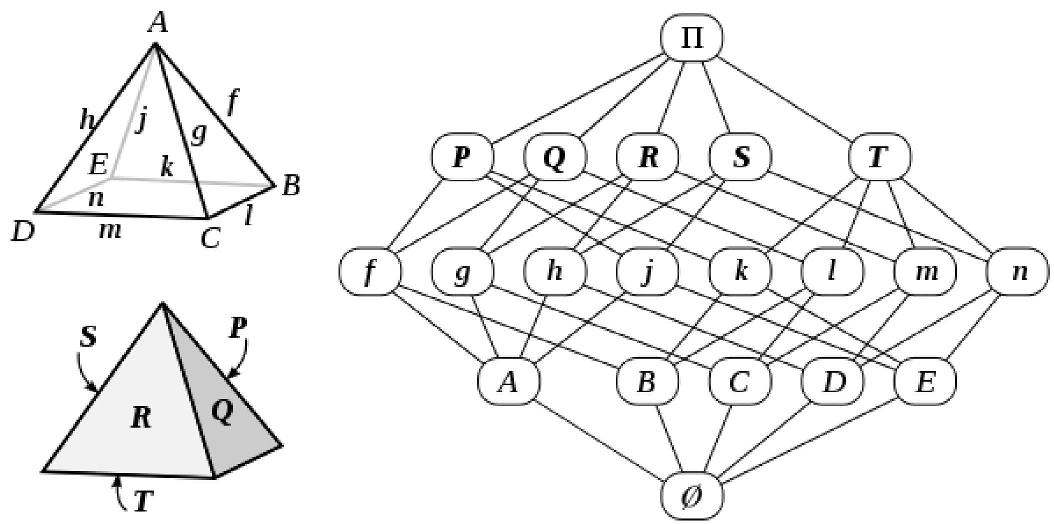
This is a much more challenging question.

Less than 10 years ago, finite chiral polytopes were known for ranks 3 and 4 only. Then some examples of rank 5 were found [by Isabel Hubard, Tomo Pisanski & MC], followed by examples of ranks 6, 7 and 8 [by Alice Devillers & MC].

At around the same time, Daniel Pellicer devised a clever construction (essentially using permutation representations of Coxeter groups) to prove that there exist finite chiral polytopes of rank n for every $n \geq 3$.

Backtracking … **Regular and chiral polytopes**

An abstract polytope $\mathcal{P}$ is a structure with the features of a geometric polytope, considered as a partially ordered set:



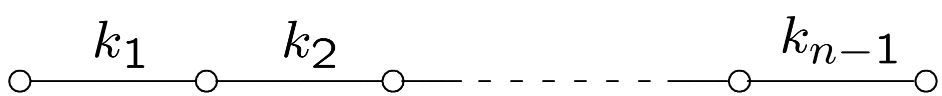
This poset must satisfy certain combinatorial conditions (namely strong connectivity and the diamond condition).

The number of intermediate layers is called the rank of $\mathcal{P}$.

Regular polytopes

An automorphism of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$. Every automorphism is uniquely determined by its effect on any given flag (maximal chain), so the number of automorphisms is bounded above by the number of flags of $\mathcal{P}$. When the upper bound is attained, we say that $\mathcal{P}$ is regular.

Also if $\mathcal{P}$ is regular, then $\mathrm{Aut}\,\mathcal{P}$ is a quotient of some 'string' Coxeter group $[\,k_1, k_2, .., k_{n-1}]$, with Coxeter/Dynkin diagram



We call $\{k_1, k_2, .., k_{n-1}\}$ the type of $\mathcal{P}$.

Chiral polytopes

Two flags are called adjacent if they differ in just one element. (In the map context, think about two faces incident with a given vertex v and edge e, or two edges incident with a given vertex v and face f, for example.)

If the automorphism group $\mathrm{Aut}\,\mathcal{P}$ of the polytope $\mathcal{P}$ has two orbits on flags, such that every two adjacent flags lie in different orbits, then $\mathcal{P}$ is said to be chiral.

In that case, $\mathrm{Aut}\,\mathcal{P}$ is a quotient of the index 2 orientation-preserving subgroup $\mathcal{C}^+$ of some Coxeter group $\mathcal{C}$, via a subgroup $\mathcal{N}$ that's normal in $\mathcal{C}^+$ but not in $\mathcal{C}$.

Constructions for chiral polytopes

In general, we can (try to) construct chiral polytopes vis their automorphism groups. We take a ‘string’ Coxeter group $\mathcal{C}$ and look for suitable quotients $G = \mathcal{C}^+/\mathcal{N}$ of the index 2 subgroup $\mathcal{C}^+$ that do not extend to quotients of $\mathcal{C}$ – this requires $\mathcal{N}$ to be normal in $\mathcal{C}^+$ but not in $\mathcal{C}$.

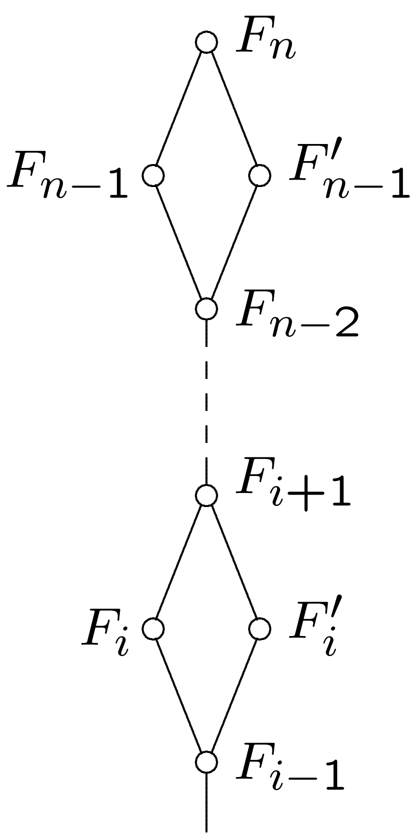
Also the quotient has to be ‘smooth’ (preserving orders of generators and certain other elements of $\mathcal{C}^+$), and satisfy an Intersection Condition on particular subgroups.

There are many ways of doing this – computational search, permutation representations, algebraic tricks, ‘mixing’, etc.

But none of these work as well for chiral polytopes as they do for regular polytopes, when the rank is greater than 3.

Drawback to inductive construction(s)

If $\mathcal{P}$ is a chiral n-polytope, then the stabilizer in $\Gamma(\mathcal{P})$ of each $(n{-}2)$-face F_{n-2} of $\mathcal{P}$ is transitive on the flags of F_{n-2}, and therefore every $(n{-}2)$-face of $\mathcal{P}$ is regular!



$\ldots$ σ swaps F_{n-1} with F'_{n-1}

$\ldots$ σ swaps F_i with F'_i

So ... construct chiral polytopes from regular!

For example, start with the n-simplex, which is a regular polytope of type $[3,\overset{n-1}{\ldots},3]$, with automorphism group S_{n+1}.

Now take a faithful permutation representation of the alternating group A_{n+1}, and extend this to a smooth permutation representation of the orientation-preserving subgroup of the $[3,\overset{n-1}{\ldots},3,m]$ Coxeter group, for some m.

Careful choice may ensure that this gives the automorphism group of a chiral polytope of rank $n+1$.

Recent theorems [proved in joint work with Isabel Hubard, Daniel Pellicer and Eugenia O'Reilly Reguiero]

- For all but finitely many positive integers n, both A_n and S_n are the automorphism groups of a chiral 4-polytope with type $\{3, 3, m\}$ for some m.

- For every $d > 3$, there are infinitely many chiral d-polytopes with type $\{3, 3, \ldots, 3, m\}$ for some m.

Special construction due to Daniel Pellicer

- Take a chiral d-polytope $\mathcal{K}$ with regular facets
- Let $\mathcal{Q}$ be the 'mixed regular cover' of $\mathcal{K}$
- Suppose/insist that $\mathcal{K}$ is 'scattered' — which means it has a flag that in $\mathcal{Q}$ is mapped far away from itself by the automorphism ρ_0
- Can then construct infinitely many chiral $(d+1)$-polytopes $\mathcal{P}$ with facets isomorphic to $\mathcal{Q}$
- Almost all such $\mathcal{P}$ are scattered, and so this construction works for all d.

Smaller/smallest chiral polytopes?

All the current approaches to constructing chiral polytopes give rise to examples that are very very large
— e.g. with A_n or S_n as automorphism group, for large n.

But there are examples that are quite small:

- Rank 3: chiral 3-polytopes with 20, 40, 42, 52, 54 flags
- Rank 4: chiral 4-polytopes with 120, 162, 192, 240 flags
- Rank 5: chiral 5-polytopes with 720, 1440 flags

Open question: What are the smallest chiral 6-polytopes?

New approach: take covers!

Let $\mathcal{P}$ be a regular or chiral polytope with ‘rotation group’ $A = \text{Aut}^+(\mathcal{P})$. If $\mathcal{Q}$ is a larger regular or chiral polytope of the same rank as $\mathcal{P}$, and its rotation group $B = \text{Aut}^+(\mathcal{Q})$ has a normal subgroup N such that B/N is isomorphic to A (in a nice way), then let’s call $\mathcal{Q}$ a cover of $\mathcal{P}$.

Auckland PhD student Wei-Juan Zhang has done some very nice work on constructing covers of given regular polytopes.

For a true ‘cover’, the type is preserved, but we can relax that condition, and this approach has been quite fruitful — e.g. for constructing infinite families of chiral polytopes of types $\{4s, 4t\}$ with $80st$ flags, or types $\{4, 4, 4s\}$ and $\{4, 4s, 4\}$ with $400s$ flags, or type $\{3s, 6, 9\}$ with $486s$ flags, and so on.

Work in progress [with PhD student Wei-Juan Zhang]

Start with a regular polytope $\mathcal{P}$ of small order compared with its rank — e.g. of type $\{4, 4, \ldots, 4\}$ or $\{4, 3, 6, \ldots, 3, 6, 3, 4\}$.

Then try to contract chiral polytopes that are 'covers' of $\mathcal{P}$ with small covering group N.

Some success to date, but the difficulty is in finding ways to make $\mathcal{P}$ and the overing group N both 'small'.

This approach should work: surprisingly, a large proportion small chiral polytopes of small rank are covers of smaller regular polytopes.

THANK YOU

THANK YOU

Abstract

Symmetry is pervasive in both nature and human culture. The notion of chirality (or 'handedness') is similarly pervasive, but less well understood. In this lecture, given in celebration of Egon Schulte and Károly Bezdek's 60th birthdays, I will talk about discrete objects that have maximum possible rotational symmetry in their class, but are not 'reflexible'. The main examples are orientably-regular maps (and the associated Riemann surfaces), and abstract polytopes. Finite chiral polytopes of large rank are notoriously difficult to construct, but I will describe some new approaches (developed in joint work with a number of people) that provide some evidence that they are not quite as rare as once thought.