Valuations on Lattice Polytopes

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joint work with Monika Ludwig

June 30, 2015

honoring Egon Schulte's and Karoly Bezdek's
60th birthday

Valuations

$\mathcal{F}$= a family of convex sets in $\mathbb{R}^n$, e.g.

- $\mathcal{C}(\mathbb{R}^n)$ =compact convex sets in $\mathbb{R}^n$
- $\mathcal{P}(\mathbb{R}^n)$ =polytopes in $\mathbb{R}^n$
- $\mathcal{P}(\mathbb{Z}^n)$ =lattice polytopes for $\mathbb{Z}^n$

$\mathbb{A}$= an Abelian semi-group, e.g.

- $\mathbb{R}$ - real valued valuation
- $\mathbb{R}^n$ - vector valued valuations
- $\mathcal{C}(\mathbb{R}^n)$ - Minkowski valuations

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$Z : \mathcal{F} \to \mathbb{A}$ is a valuation if

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L)$$

for any $K, L \in \mathcal{F}$ satisfying $K \cap L \in \mathcal{F}$ and $K \cup L \in \mathcal{F}$.

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Origin Dehn's solutions of Hilbert's scissors congruency problem

Examples of Valuations and Group actions

Support function $h : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$

- If $K, L, K \cap L, K \cup L \in \mathcal{C}(\mathbb{R}^n)$, then $h_{K \cap L} + h_{K \cup L} = h_K + h_L$

Intrinsic volumes $V_i : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$, $i = 0, \ldots, n$ (rigid motion invariant)

- $V_0(K) = 1$ (Euler characteristic)
- $V_n(K) =$volume
- $V_i(K)$ - "i-dimensional mean projection", $i = 1, \ldots, n-1$

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Minkowski valuations $Z : \mathcal{C}(\mathbb{R}^n) \to \mathcal{C}(\mathbb{R}^n)$

Difference body $DK = K - K$

- $\mathrm{SL}(n, \mathbb{R})$ equivariant, translation invariant

Projection body $\Pi : \mathcal{C}(\mathbb{R}^n) \to \mathcal{C}(\mathbb{R}^n)$, $h_{\Pi K,u} = V_{n-1}(K|u^\perp)$, $u \in S^{n-1}$, $K|u^\perp$ is the projection into $u^\perp$

- $\mathrm{SL}(n, \mathbb{R})$ contravariant, translation invariant

The Hadwiger Classification Theorem, 1952

Theorem

$Z : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}$ is rigid motion invariant and continous valuation iff there exist $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$Z(K) = \sum_{i=0}^{n} c_i V_i(K)$$

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Remark

- Alesker (2000) characterized $\mathrm{SO}(n, \mathbb{R})$ invariant and polynomial valuations

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Open problem

- Characterize rigid motion invariant valuations $Z : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$

Polynomial valuations

Definition $Z : \mathcal{F} \to \mathbb{R}^k$ is a polynomial valuation of degree at most d if

$$Z(K + x) = Z(K) + \theta(K, x) \quad \text{for any } K \in \mathcal{F}$$

where $\theta(K, x)$ is a polynomial of degree at most d in x where $x \in \mathbb{R}^n$ for $\mathcal{F} = \mathcal{C}(\mathbb{R}^n), \mathcal{P}(\mathbb{R}^n)$, and $x \in \mathbb{Z}^n$ for $\mathcal{F} = \mathcal{P}(\mathbb{Z}^n)$

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Examples $\varphi : \mathbb{R}^n \to \mathbb{R}^k$ polynomial of degree d

$$\text{either} \quad Z(K) = \int_K \varphi(y)\, dy \quad \text{or} \quad Z(K) = \sum_{y \in K \cap \mathbb{Z}^n} \varphi(y)$$

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Theorem (McMullen,Khovanski,Pukhilov)

If Z is a polynomial valuation of degree at most d, and $\lambda \in \mathbb{N}$, then

$$Z(\lambda P) = \sum_{i=0}^{n+d} Z_i(P)\lambda^i$$

where Z_i homogeneous valuation valuation of degree i, and Z_1 is Minkowski additive; namely, $Z_1(K+L) = Z_1(K) + Z_1(L)$.

Steiner point

Definition $s : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}^n$

$$s(K) = \frac{1}{V_n(B^n)} \int_{S^{n-1}} u h_K(u)\, du$$

where B^n is the Euclidean unit ball.

Theorem (Schneider (1971))

$Z : \mathcal{C}(\mathbb{R}^n) \to \mathbb{R}^n$ is Minkowski additive, continuous and rigid motion equivariant valuation iff $Z = s$.

$\mathrm{SL}(n,\mathbb{R})$ interwining Minkowski valuations

Theorem (Ludwig (2005))

Let $n \geq 2$, and let $Z : \mathcal{P}(\mathbb{R}^n) \to \mathcal{C}(\mathbb{R}^n)$ be Minkowski valuation.

- *Z is $SL(n,\mathbb{R})$ equivariant and translation invariant iff there exists $\alpha \geq 0$ such that $Z(K) = \alpha(K - K)$.*
- *Z is $SL(n,\mathbb{R})$ contravariant and translation invariant iff there exists $\alpha \geq 0$ such that $Z(K) = \alpha \Pi K$.*

Remark Z is an $\mathrm{SL}(n,\mathbb{R})$ contravariant means

$$Z(\Phi K) = \Phi^{-t} Z(K) \quad \text{for} \quad \Phi \in \mathrm{SL}(n,\mathbb{R})$$

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Remark Haberl, Parapatits (2014) characterized $\mathrm{SL}(n,\mathbb{R})$ equivariant Minkowski valuations and $\mathrm{SL}(n,\mathbb{R})$ contravariant Minkowski valuations (without translation invariance)

Valuations on lattice polytopes

Definition Lattice point enumerator

$$G(K) = \#(K \cap \mathbb{Z}^n) \quad \text{for} \quad K \in \mathcal{P}(\mathbb{Z}^n).$$

Theorem (Ehrhart (1967))

For $K \in \mathcal{P}(\mathbb{Z}^n)$ *and* $\lambda \in \mathbb{N}$, $G(\lambda K) = \sum_{i=0}^{n} G_i(K)\lambda^i$.

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For $K \in \mathcal{P}(\mathbb{Z}^n)$ and $\lambda \in \mathbb{N}$, $G(\lambda K) = \sum_{i=0}^{n} G_i(K)\lambda^i$.

Theorem (Betke, Kneser (1985))

$Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ is an $SL(n, \mathbb{Z})$ and translation invariant valuation iff there exist $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$ such that

$$Z(K) = \sum_{i=0}^{n} \alpha_i G_i(K).$$

discrete Steiner point

Definition discrete moment vector $m : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$

$$m(K) = \sum \{y : y \in K \cap \mathbb{Z}^n\}$$

Remark $m(K + x) = m(K) + G(K)x$, and hence

$$m(\lambda K) = \sum_{i=0}^{n+1} m_i(K)\lambda^i \quad \text{for } \lambda \in \mathbb{N}$$

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Definition $\sigma = m_1$ the discrete Steiner point (Minkowski additive)

Remark

- $\sigma(K)$ is the centroid if K is a unimodular simplex or centrally symmetric

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Theorem (K.J. Boroczky, M. Ludwig)

$Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is $SL(n, \mathbb{Z})$ and translation equivariant valuation iff $Z = \sigma$.

$\mathrm{SL}(n,\mathbb{Z})$ interwining Minkowski valuations on lattice polytopes

Theorem (K.J. Boroczky, M. Ludwig)

Let $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{C}(\mathbb{R}^n)$.

- *Z is an $SL(n,\mathbb{Z})$ equivariant and translation invariant Minkowski valuation for $n \geq 2$ iff there exist $\alpha, \beta \geq 0$ such that*

$$Z(K) = \alpha(K - \sigma(K)) + \beta((-K) - \sigma(-K)).$$

- *Z is an $SL(n,\mathbb{Z})$ contravariant and translation invariant Minkowski valuation for $n \geq 3$ iff there exists $\alpha \geq 0$ such that $Z(K) = \alpha \Pi K$.*

Many more beautiful theorems to Egon and Karoly