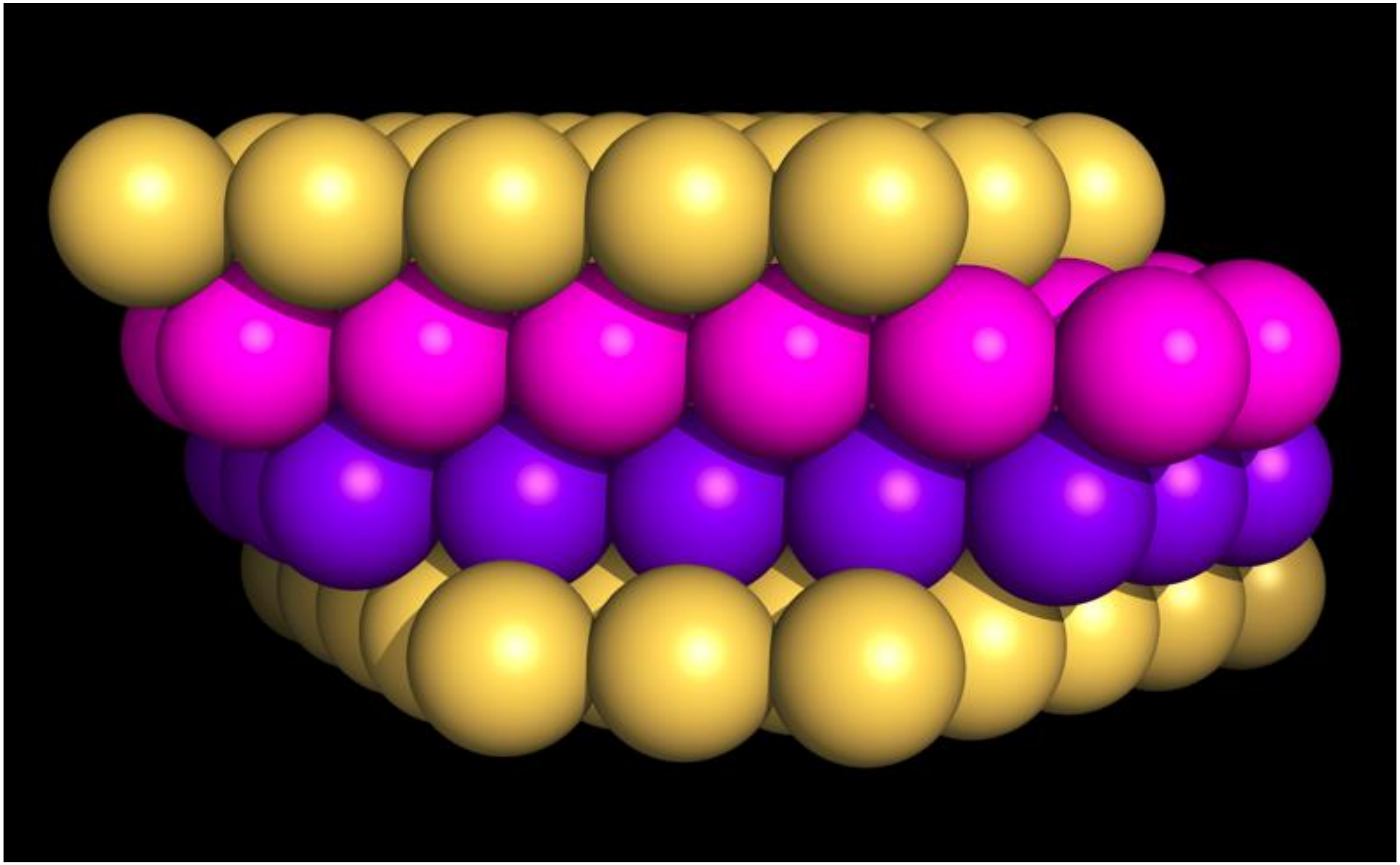
12-neighbour packings of unit balls in $\mathbb{E}^3$

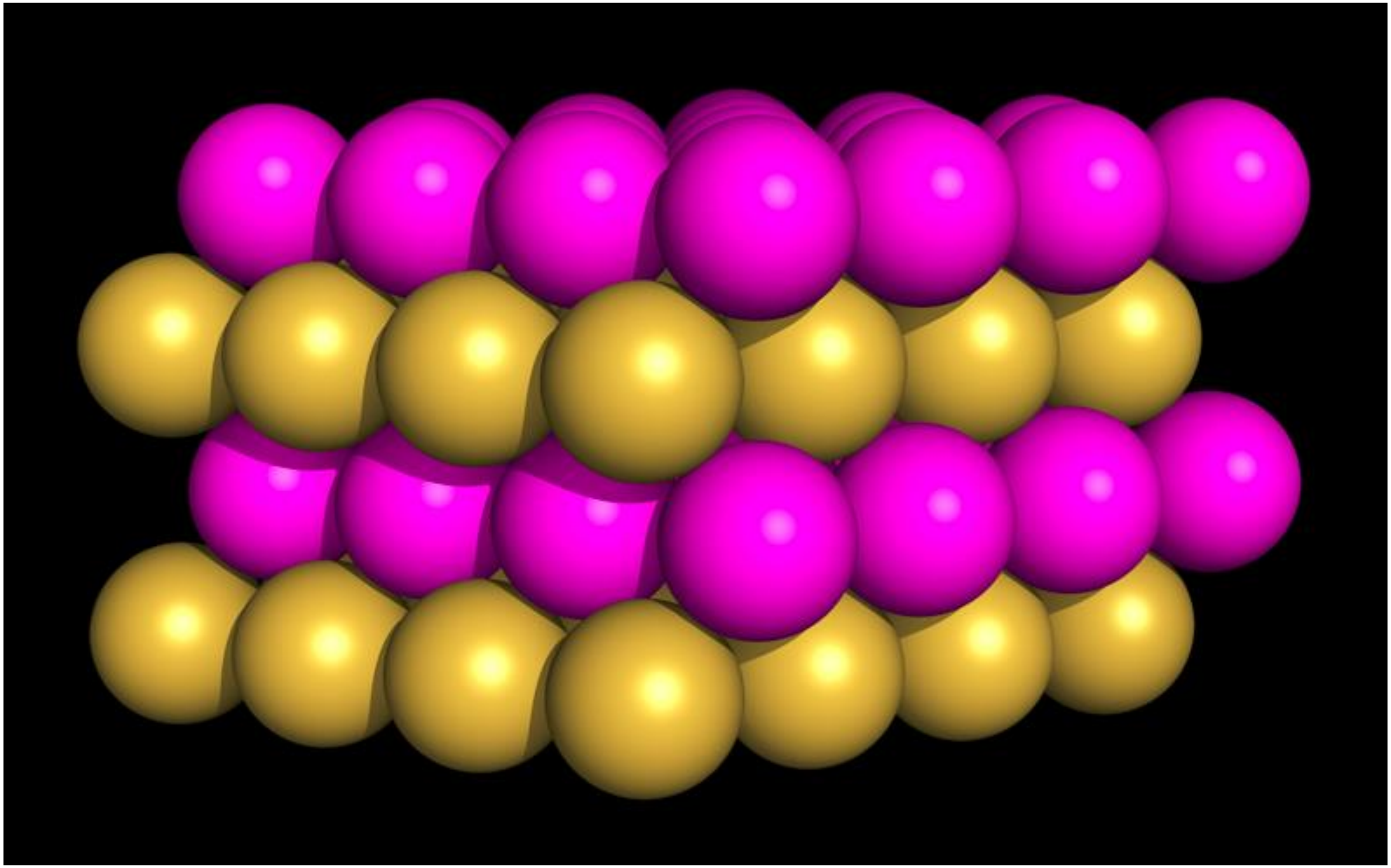
K. Böröczky and L. Szabó

A packing of unit balls in $\mathbb{E}^3$ is said to be a 12-neighbour packing if each ball is touched by 12 others.

Face-centred cubic packing



Hexagonal-close packing



Fejes Tóth's conjecture [1969, 1989].

Any 12-neighbour packing of unit balls in $\mathbb{E}^3$ is composed of hexagonal layers.

T. C. Hales [2012].

A proof of Fejes Tóth's conjecture on sphere packings with kissing number twelve.

eprint arXiv:1209.6043.

Tammes problem for 13 points.

What is the maximum number a_{13} with the property that one can place 13 points on $\mathbb{S}^2$ so that the spherical distance between any two different points is at least a_{13}?

Theorem (Musin, Tarasov [2012]).

$a_{13} = 57.13670307\ldots°$.

Set $a_0 = 57.13670309° > a_{13}$.

Proposition.

The distance between the centres of any two non-neighbouring balls in a 12-neighbour packing of unit balls in $\mathbb{E}^3$ is at least

$$4\sin\left(180°\left(\frac{60°}{a_0}-\frac{5}{6}\right)\right) = 2.51838585\ldots$$

Lemma (spherical geometry).

Let C be a point on $\mathbb{S}^2$ and let $0 < \lambda < 1$.

Let C^* be the antipodal of C on $\mathbb{S}^2$.

Let P and Q be two points on $\mathbb{S}^2$ different from C^* and let P' and Q' denote the points of the segments CP and CQ, respectively, on $\mathbb{S}^2$ for which

$$CP' = \lambda \cdot CP \text{ and } CQ' = \lambda \cdot CQ.$$

Then

$$P'Q' \geq \lambda \cdot PQ.$$

Straightforward calculation shows that the angle of the triangle with side lengths 2, 2 and $2.51838585\ldots$ opposite to its longest side is $78.04071344\ldots°$.

Set $b_0 = 78.04071344° < 78.04071344\ldots°$.

Set $r_0 = (180° - b_0)/2 = 50.97964328°$.

Proposition.

Let B_0 be a ball in a 12-neighbour packing of unit balls in $\mathbb{E}^3$ and let $\mathcal{P}$ denote the set of the points at which its 12 neighbours touch B_0.

Then, regarding $\mathcal{P}$ as a point set on $\mathbb{S}^2$,

- the distance between any two different points of $\mathcal{P}$ is either $60°$ or at least b_0,
- the radius of any circle whose interior does not contain any point of $\mathcal{P}$ is smaller than r_0.

Let $\mathcal{C}$ be the configuration of the touching points on a given ball with its neighbours in the face-centred cubic packing.

Let $\mathcal{C}'$ be the configuration of the touching points on a given ball with its neighbours in the hexagonal-close packing.

Theorem.

Let $\mathcal{P}$ be a set of 12 points on $\mathbb{S}^2$ such that

- the distance between any two different points of $\mathcal{P}$ is either $60°$ or at least b_0,
- the radius of any circle whose interior does not contain any point of $\mathcal{P}$ is smaller than r_0.

Then $\mathcal{P}$ is congruent to either $\mathcal{C}$ or $\mathcal{C}'$.

Outline of the proof.

Let $\mathcal{D}$ be a Delone triangulation of $\mathcal{P}$.

For each $i = 0,1,2,3,$ a triangle face of $\mathcal{D}$ will be called of type i if it has i sides of length at least b_0 and $3 - i$ sides of length 60°.

Proposition.

1) There are 12 vertices, 30 edges and 20 triangle faces in $\mathcal{D}$.
2) For each $i = 0,1,2,3$, the area of a triangle face of type i of $\mathcal{D}$ is greater than or equal to the area of a triangle with i sides of length b_0 and $3 - i$ sides of length $60°$.
3) Each vertex of $\mathcal{D}$ is a common vertex of at most two triangle faces of type 0 of $\mathcal{D}$.

4) There are at most 8 triangle faces of type 0 in $\mathcal{D}$.
5) There is no triangle face of type 3 in $\mathcal{D}$.
6) There is at most 1 triangle face of type 2 in $\mathcal{D}$.

Let $\mathcal{A}$ be the subgraph of $\mathcal{D}$ formed by the edges (and their endpoints) of length $60°$ of $\mathcal{D}$.

Proposition.

1) Each vertex of $\mathcal{D}$ is a vertex of $\mathcal{A}$ as well, i.e. the number of vertices of $\mathcal{A}$ is 12.
2) The number of edges of $\mathcal{A}$ is 24.
3) Each vertex of $\mathcal{A}$ is of degree 4.
4) The number of faces of $\mathcal{A}$ is 14.
5) Eight faces of $\mathcal{A}$ are regular triangles and six faces of $\mathcal{A}$ are squares.
6) Each vertex of $\mathcal{A}$ is adjacent to two regular triangle faces and two square faces of $\mathcal{A}$.

Case 1.

Each edge of $\mathcal{A}$ is adjacent to one regular triangle face and one square face of $\mathcal{A}$.

Then $\mathcal{A}$ is congruent to the Archimedean tiling (3,4,3,4).

Thus $\mathcal{P}$ is congruent to $\mathcal{C}$.

Case 2.

There is an edge of $\mathcal{A}$ adjacent to either two regular triangle faces or two square faces of $\mathcal{A}$.

Then the great circle incident to this edge consists of six such edges of $\mathcal{A}$, alternately adjacent to either two regular triangle faces or two square faces of $\mathcal{A}$.

Thus $\mathcal{P}$ is congruent to $\mathcal{C}'$.

This completes the proof of the theorem.

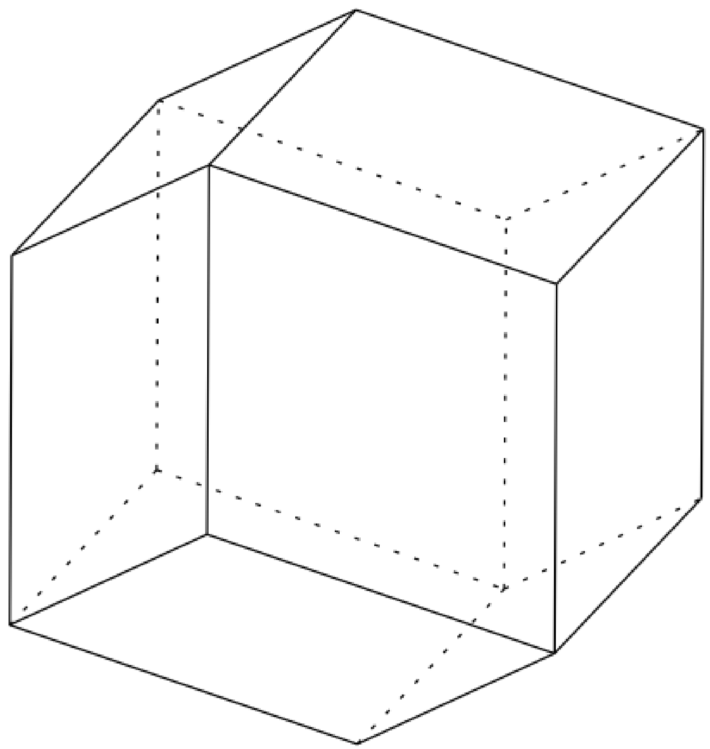
Proposition.

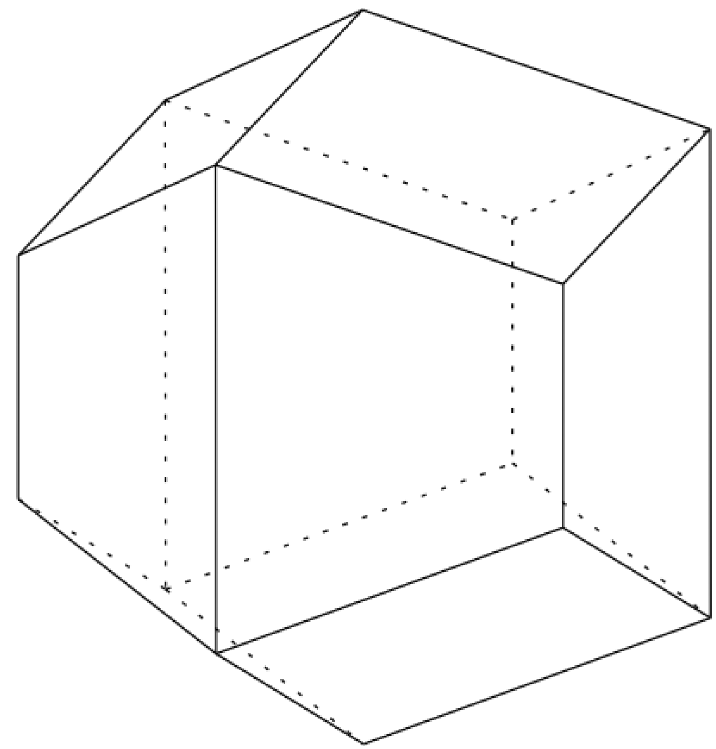
Let $\mathcal{B}$ be a 12-neighbour packing of unit balls in $\mathbb{E}^3$ and consider the Dirichlet-Voronoi cell decomposition of $\mathbb{E}^3$ associated to $\mathcal{B}$. Let B_0 be a unit ball in $\mathcal{B}$.

Then the Dirichlet-Voronoi cell of B_0 is

- either a rhombic dodecahedron
- or a trapezo-rhombic dodecahedron

circumscribed about B_0.





Case 1.

Each Dirichlet-Voronoi cell is a rhombic dodecahedron.

Then $\mathcal{B}$ is uniquely determined and coincides with the densest lattice packing of unit balls which consists of parallel hexagonal layers.

Case 2.

There is a Dirichlet-Voronoi cell which is a trapezo-rhombic dodecahedron.

Then there is a hexagonal layer $\mathcal{L}$ in the Dirichlet-Voronoi cell decomposition consisting of trapezo-rhombic dodecahedra each of which is adjacent to six others along their common trapezoid faces.

Now, the Dirichlet-Voronoi cells adjacent to $\mathcal{L}$ along their common rhombus faces on the same side of $\mathcal{L}$ form

- either a hexagonal layer of rhombic dodecahedra
- or a hexagonal layer of trapezo-rhombic dodecahedra

parallel to $\mathcal{L}$.

By repeated applications of this argument, one obtains that the Dirichlet-Voronoi cell decomposition consists of parallel hexagonal layers of rhombic dodecahedra and trapezo-rhombic dodecahedra.

This implies that $\mathcal{B}$ consists of parallel hexagonal layers of unit balls in this case, too.

Thank you for your attention!