Existence of closed billiard trajectories in “acute-angled” bodies

Alexey Balitskiy [1]
based on joint work with Arseniy Akopyan

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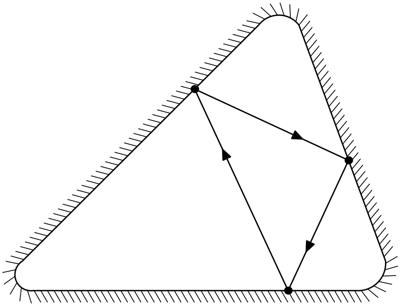
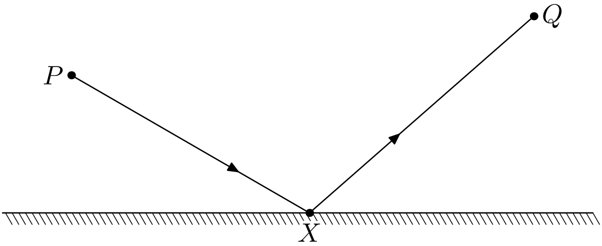


Reflection rule



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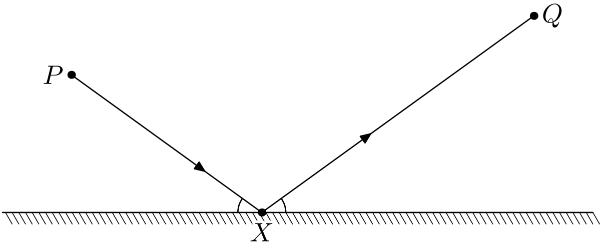
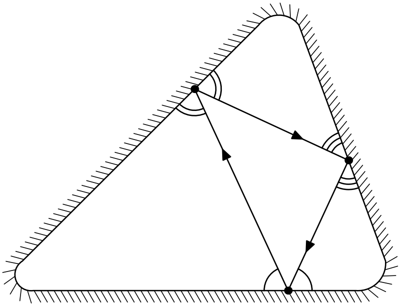
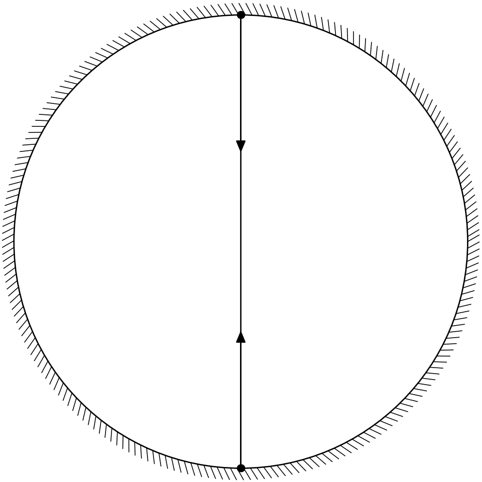


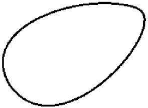
$|PX| + |QX|$ has a local extremum at this point.





Billiard

Generally, we consider a convex body $K \subset \mathbb{R}^n$ as a billiard table.



We are interested in existence of *classical* (i.e. passing only through smooth points of the boundary ∂K) closed billiard trajectories in the body K.
We say m-bouncing, or m-periodic, about closed trajectory with m boundary reflections.

Existence of trajectories in smooth bodies

Here K is smooth and strictly convex.

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- [M. Farber, S. Tabachnikov, 2002] If $K \subset \mathbb{R}^n$, $n \geq 3$, m is odd, then there are at least $\lfloor \log_2(m-1) \rfloor + n - 1$ distinct closed billiard trajectories with m bounces. For generic K there are at least $(m-1)(n-1)$ such trajectories.

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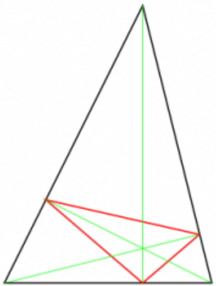
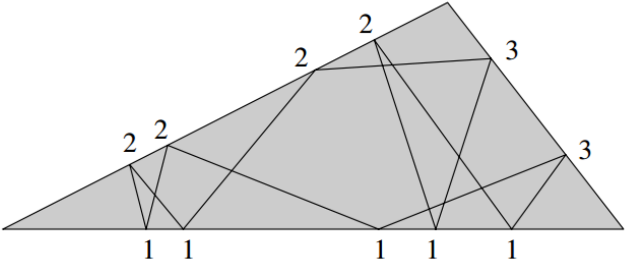
- [R. Karasev, 2008] If $K \subset \mathbb{R}^n$, $n \geq 3$, m is odd prime, then there are at least $(m-1)(n-2)+2$ distinct closed billiard trajectories with m bounces.

Existence of trajectories in non-smooth bodies



In an acute triangle there is a classical closed billiard trajectory.
The idea goes back to H. Schwarz (1890)

Existence of trajectories in non-smooth bodies



[R. E. Schwartz, 2009] In a triangle with angles $\leq 100^\circ$ there is a classical closed billiard trajectory.

Existence of trajectories in non-smooth bodies

Definition

We say that a non-smooth point $q \in \partial K$ satisfies the *acuteness condition* if the tangent cone $T_K(q)$ can be represented as the orthogonal product $T_K(q) = F \times T^k$, where T^k is a k-dimensional cone with property that for all points $a, b \in T^k$ the inequality $\widehat{aqb} < \pi/2$ holds, and F is an $(n-k)$-dimensional linear subspace orthogonal to T^k.

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Definition

If all non-smooth points of ∂K satisfy the above acuteness condition we call K an *acute body*.

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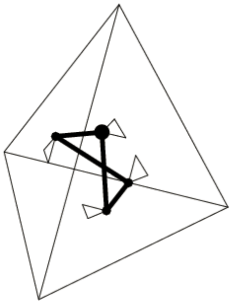
Definition

If all non-smooth points of ∂K satisfy the above acuteness condition we call K an *acute body*.

Theorem (Akopyan, B., 2015+)

In an acute convex body $K \in \mathbb{R}^n$ there exists a closed classical billiard trajectory with no more than $n+1$ bounces.

Existence of trajectories in non-smooth bodies

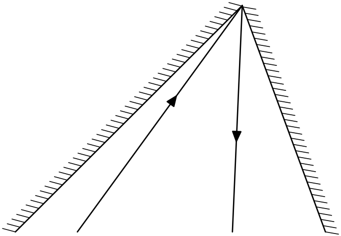


Corollary (Akopyan, B., 2015+)

In a simplex with all acute dihedral angles (e.g., a simplex close to regular) there exists a closed classical billiard trajectory with $n + 1$ *bounces.*

Bounce at corners

Generalized (in contrast with classical) trajectories CAN pass through non-smooth points.



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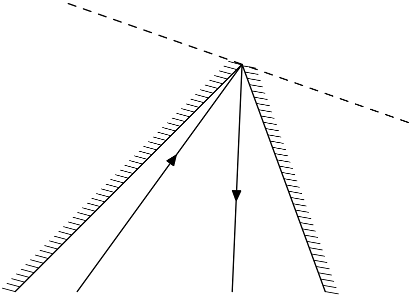
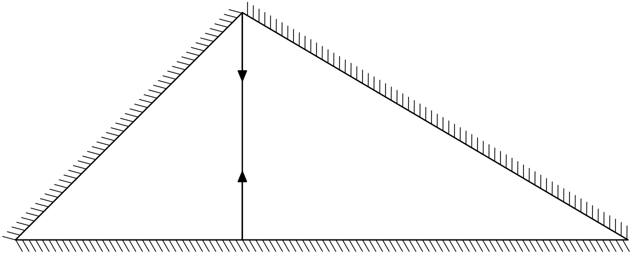


Bounce at corners

That is OK.



Question

What can be said about the shortest closed billiard trajectory in a convex body K?

Theorem (Károly Bezdek and Daniel Bezdek, 2009)

Let K be a convex body in $\mathbb{R}^n$. Then any of the shortest (Euclidean) generalized closed billiard trajectories in K is of period at most $n+1$.

Geom Dedicata (2009) 141:197–206
DOI 10.1007/s10711-009-9353-6

ORIGINAL PAPER

Shortest billiard trajectories

Dániel Bezdek · Károly Bezdek

Received: 25 February 2008 / Accepted: 8 January 2009 / Published online: 29 January 2009

Abstract In this paper we prove that any convex body of the d-dimensional Euclidean space ($d \geq 2$) possesses at least one shortest generalized billiard trajectory moreover, any of its shortest generalized billiard trajectories is of period at most $d + 1$. Actually, in the Euclidean plane we improve this theorem as follows. A disk-polygon with parameter $r > 0$ is simply the intersection of finitely many (closed) circular disks of radii r, called generating disks, having some interior point in common in the Euclidean plane. Also, we say that a disk-polygon with parameter $r > 0$ is a fat disk-polygon if the pairwise distances between the centers of its generating disks are at most r. We prove that any of the shortest generalized billiard trajectories of an arbitrary fat disk-polygon is a 2-periodic one. Also, we give a proof of the analogue result for ε-rounded disk-polygons obtained from fat disk-polygons by rounding them off using circular disks of radii $\varepsilon > 0$. Our theorems give partial answers to the very recent question raised by S. Zelditch on characterizing convex bodies whose shortest periodic billiard trajectories are of period 2.

Keywords (fat) Disk-polygon · (generalized) Billiard trajectory · Shortest (generalized) billiard trajectory

Mathematics Subject Classification (2000) 52A40 · 52C99

Corollary

Any shortest billiard trajectory in the body of constant width 1 *in the plane has period* 2.

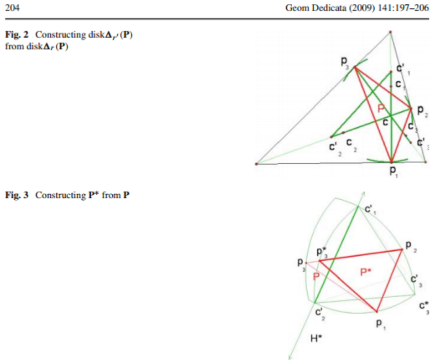

204 Geom Dedicata (2009) 141:197–206

Fig. 2 Constructing disk$\Delta_{r'}$(**P**) from diskΔ_r(**P**)

Fig. 3 Constructing **P*** from **P**

Bezdeks' trajectories

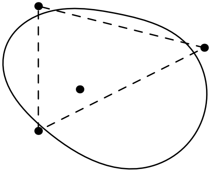
Let V be an n-dimensional vector space, $K \subset V$, and define

$$\mathcal{P}_m(K) = \{(q_1, \dots, q_m) : \\ \{q_1, \dots, q_m\} \text{ does not fit into } \alpha K + t \text{ with } \alpha \in (0,1),\ t \in V\}.$$

Define the length of the closed polygonal line

$$\ell\{q_1, \dots, q_m\} = \sum_{i=1}^{m} |q_{i+1} - q_i|,$$

where indices are always modulo m.

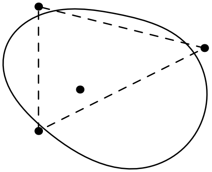


Theorem

For a convex body $K \in V$, the length of the shortest generalized closed billiard trajectory in K equals

$$\xi(K) = \min_{m \geq 2} \min_{P \in \mathcal{P}_m(K)} \ell(P).$$

Moreover, the minimum is attained at $m \leq n+1$.



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- We consider minimizer delivering $\xi(K)$ and show that it can be translated to obtain generalized billiard trajectory.
- Finally, any billiard trajectory indeed cannot be translated into the interior of K.

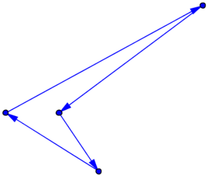
Particular case of Bezdeks' lemma

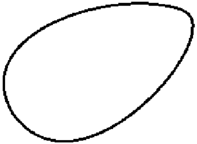
Lemma (K. Bezdek, D. Bezdek, 2009)

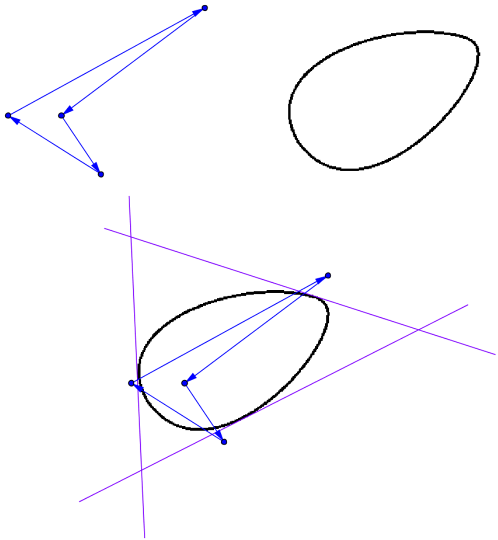
Suppose the points $q_1, \dots, q_m$ *satisfy the following condition: There exist affine halfspaces* $H_1^+, \dots, H_m^+$ *with outer normals* $n_1, \dots, n_m$*, such that*

1. $q_i \in \partial H_i^+$ *for* $i = 1, \dots, m$*;*
2. $K \subset H_i^+$ *for* $i = 1, \dots, m$*;*
3. $0 \in \operatorname{conv}\{n_1, \dots, n_m\}$.

Then the polygonal line with vertices $q_1, \dots, q_m$ *(and maybe with some other vertices) cannot be translated into* $\operatorname{int} K$.







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From the reflection rule for a billiard trajectory $\{q_1, \dots, q_m\}$ we have

$$p_{i+1} - p_i = -\lambda_i n_K(q_i), \quad \lambda_i > 0.$$

Here we denote the momenta by $p_i = \dfrac{q_i - q_{i-1}}{|q_i - q_{i-1}|}$.

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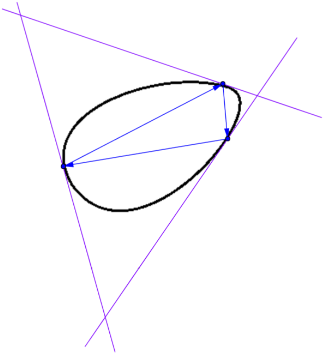

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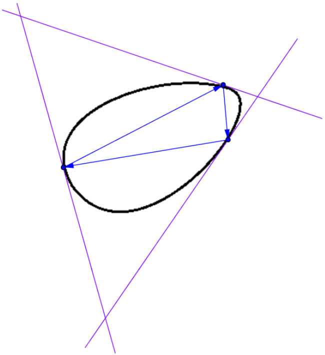

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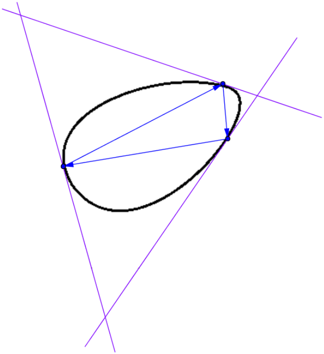

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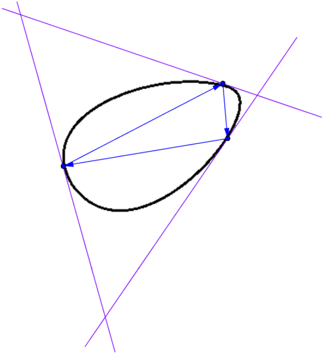

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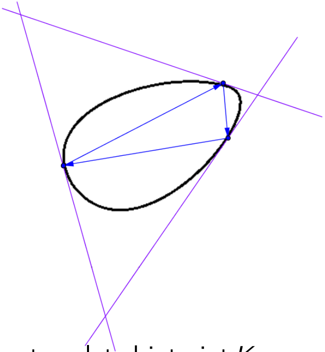

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The lemma implies that the set $\{q_1, \dots, q_m\}$ cannot be translated into int K.

Main theorem

Definition

We say that a non-smooth point $q \in \partial K$ satisfies the *acuteness condition* if the tangent cone $T_K(q)$ can be represented as the orthogonal product $T_K(q) = F \times T^k$, where T^k is a k-dimensional cone with property that for all points $a, b \in T^k$ the inequality $\widehat{aqb} < \pi/2$ holds, and F is an $(n-k)$-dimensional linear subspace orthogonal to T^k.

Definition

If all non-smooth points of ∂K satisfy the above acuteness condition we call K an *acute body*.

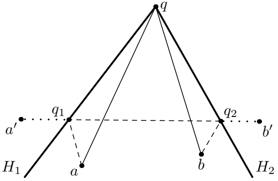
Theorem (Akopyan, B., 2015+)

In an acute convex body $K \in \mathbb{R}^n$ there exists a closed classical billiard trajectory with no more than $n+1$ bounces.

Two-dimensional case

Theorem

In an acute convex disc $K \in \mathbb{R}^2$ there exists a closed classical billiard trajectory with 2 or 3 bounces.



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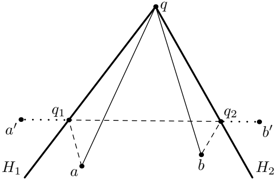

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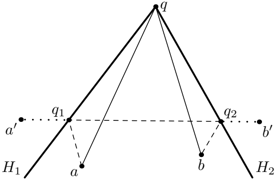

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Reflect a and b in support lines H_1 and H_2 respectively and obtain point a' and b'.

$\angle(H_1, H_2) < \frac{\pi}{2} \quad \Rightarrow \quad \angle a'qb' < \pi$.

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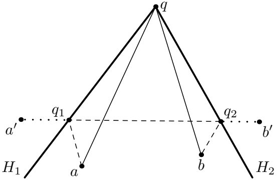

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$\angle(H_1, H_2) < \frac{\pi}{2} \quad \Rightarrow \quad \angle a'qb' < \pi$.
$|aq_1| + |q_1q_2| + |q_2b| = |a'q_1| + |q_1q_2| + |q_2b'| = |a'b'| < |a'q| + |qb'| = |aq| + |qb|$.

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Corollary

In a simplex with all acute dihedral angles (e.g., a simplex close to regular) there exists a closed classical billiard trajectory with $n+1$ *bounces.*

Theorem

If the shortest closed generalized trajectory in $K \subset \mathbb{R}^n$ *has precisely* $n+1$ *bounces, then it is classical.*

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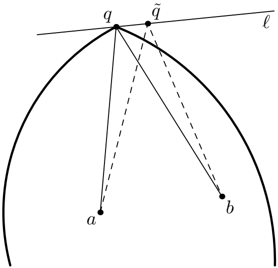
Assume the contrary: $a \to q \to b$ is a fragment of the shortest trajectory near non-smooth point $q \in \partial K$.
Note that a, b, q do not lie on the same line.



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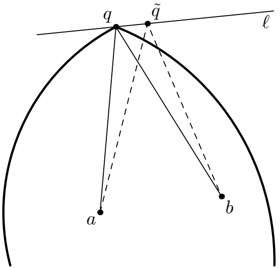

Consider the support line ℓ orthogonal to bisector of $\widehat{aqb}$ at the point q. It can be slightly rotated remaining support at the point q. We find point $\tilde{q} \in \ell$ such that $|\tilde{q} - a| = |\tilde{q} - b|$.

Theorem

If the shortest closed generalized trajectory in $K \subset \mathbb{R}^n$ has precisely $n+1$ bounces, then it is classical.

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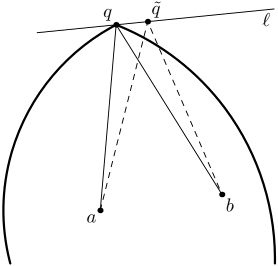

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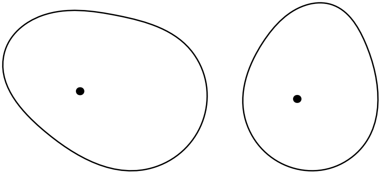
$|\tilde{q} - a| + |\tilde{q} - b| < |q - a| + |q - b|$.

Non-Euclidean billiards

We use *possibly non-standard* notation for a norm $\|\cdot\|_T$ with T lying in the dual space: $\|q\|_T = \max\limits_{p\in T}\langle p, q\rangle$.



$K \subset V$ $T \subset V^*$

In other words, $T^\circ = \{q \in V : \langle p, q\rangle \leq 1 \ \forall p \in T\}$ is the unit body of the norm $\|\cdot\|_T$.

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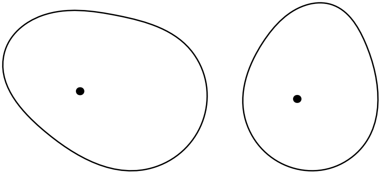
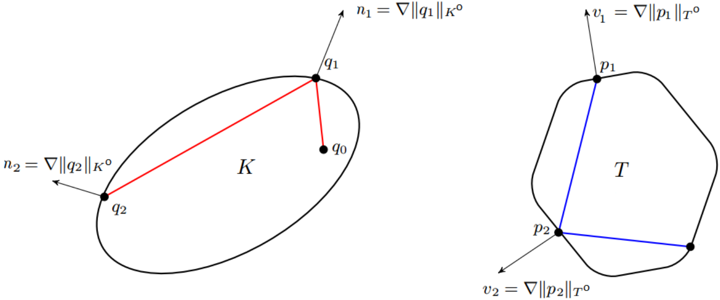


$K \subset V$ $\quad$ $T \subset V^*$

In other words, $T^\circ = \{q \in V : \langle p, q\rangle \leq 1\ \forall p \in T\}$ is the unit body of the norm $\|\cdot\|_T$.
Note that T need not be symmetric, so in general $\|q\| \neq \|-q\|$.

Non-Euclidean billiards

Billiards are defined as before.



The reflection rule: $p_2 - p_1 = -\lambda n_K(q_1), \quad \lambda > 0.$

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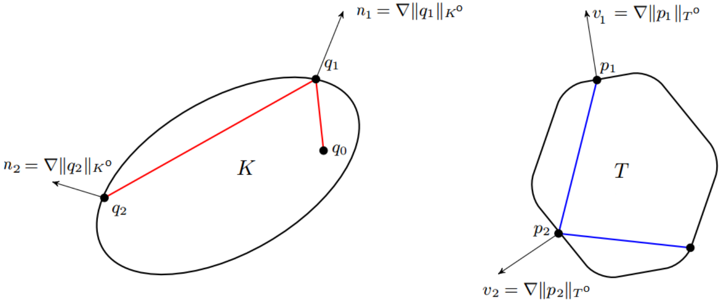

The reflection rule: $p_2 - p_1 = -\lambda n_K(q_1), \quad \lambda > 0.$

Bezdeks' characterization still holds!

Non-Euclidean billiards

Theorem

Suppose the length is measured using the norm with strictly convex unit body T° such that T is strictly convex too (in other words, T is smooth and strictly convex).
If the shortest closed generalized trajectory in $K \subset \mathbb{R}^n$ has $n+1$ bounces, then it is classical, that is, it does not pass through non-smooth points of ∂K.

Thank you for your attention!