Flag-transitive biplanes, what do we know?

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Definition

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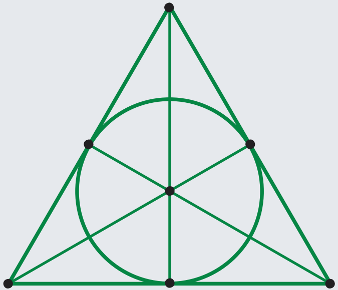
- Every (unordered) pair of points is incident with exactly 2 blocks.

Definitions & Examples
Fano plane

Nonexample



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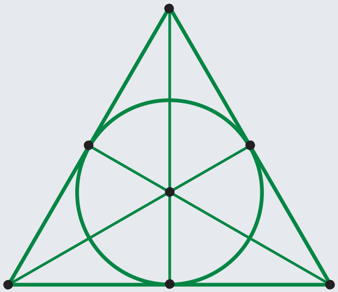
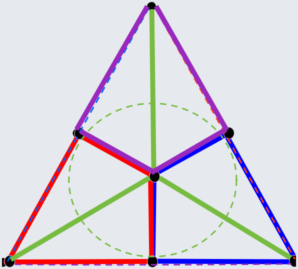


Ceci n'est pas un biplan.

Definitions & Examples

Fano's complement

Example



$(7, 4, 2)$-biplane

Definitions & Examples

Flag-transitivity

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- A flag in a biplane D is an incident point-block ordered pair.
- A group $G \leq \mathrm{Aut}(D)$ is flag-transitive if it has exactly one orbit on the set of flags of D.

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Such a design has $(n + 1)(n^2 + n + 1) = vk$ flags.

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Projective planes

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No non-Desarguesian examples are known, and it has been long conjectured that there are none (Higman and McLaughlin).

Classification
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Theorem (K. Thas, D. Zagier, 2008)

If D is a $(n^2 + n + 1, n + 1, 1)$-symmetric design and $n^2 + n + 1 < 4 \times 10^{22}$ with $n^2 + n + 1 = p$ prime, then D is Desarguesian.

Conjecture

For any given value of $\lambda > 1$, there are finitely many (v, k, λ)-symmetric designs.

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There are no biplanes with $k = 7, 8, 10$, or 12.

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4. Otherwise it is primitive, (G leaves no non-trivial partition of P invariant).

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- $k \leq \lambda(\lambda - 2)$.

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- In particular, for biplanes, the only admissible parameters are $(16, 6, 2)$.
- There are two (16,6,2) biplanes arising from difference sets in $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_2^4$, both admitting a flag-transitive, imprimitive automorphism group G with $G_0 \cong S_4$.

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If D is a (v, k, λ)-symmetric design with $(2 \leq \lambda \leq 4)$ *admitting a flag-transitive, primitive automorphism group G, then G is of almost simple or affine type.*

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- *(11,5,2) and $G \leq PSL_2(11)$ (this is a Hadamard design).*

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The (37,9,2) biplane is an example of the one-dimensional affine case.

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- $G \leq A\Gamma L_1(q)$ *for some odd prime power* q.

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Lemma (EOR, 2005)

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If D is a $(p^r, k, 2)$-biplane with p is prime, then either $p^r = 16$ or p is odd.

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Lemma (P. Cameron, 2002)

Let G be an affine automorphism group of a $(v, k, 2)$-biplane, with $v = p^r$, p prime. Suppose $G = T.H$, where T is the translation group of $V(r, p)$ acting regularly on the points of the biplane, and $H \leq GL(r, p)$. If p is odd then $|G|$ is odd.

Theorem (EOR, C. Praeger, 2009)

Let D be a $(q, k, 2)$-biplane with $q = p^r$ an odd prime power admitting a flag-transitive automorphism group of affine type $G \leq A\Gamma L(1, q)$. Then p does not divide $|G_0|$, and G acts regularly on the flags of D.

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If D is as above, then $G \leq AGL(1, q)$ (with $|G| = vk$, $G_0 \leq GL(1, q)$ and $|G_0| = k$).

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- and then...

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- They prove the conjecture for $p < 10000000$.

Very likely corollary

If D is a flag-transitive biplane with p points and $37 < p$ prime, then $10000000 < p$.

Köszönöm!

Thank you!