International Conference
on

Geometry and Symmetry

Abstracts

Department of Mathematics
Faculty of Information Technology
University of Pannonia
Veszprém, Hungary

29 June – 3 July, 2015
Veszprém, Hungary

On polyhedra induced by point sets and on their triangulations

András Bezdek
Auburn University, USA and Alfréd Rényi Institute, Hungary

It is well known that finite point sets (not all points lying on a line) can be polygonized, meaning that one can construct polygons such that the vertices of the polygons coincide with the given point set. It is also known that planar polygons can be triangulated using only diagonals of the polygons. Stepping one dimension higher one might be interested in polyhedronizations of finite point sets so that the resulting polyhedrons can be partitioned with tetrahedra in a face-to-face fashion without introducing new vertices. Such partitions in 3-space are also called triangulations. Schönhardt (1927), Bagemihl (1948), Kuperberg (2011) and others constructed special polyhedra in such a way that clever one line geometric reasons imply nontriangulability. More complicated constructions were given by Rambau (2011) and Bezdek and Carrigan (2014). In this talk we survey the algorithms used for polyhedronization of 3D point sets. We also describe new algorithms. This is an ongoing research with Braxton Carrigan and Osman Yardimci.

12-neighbour packings of unit balls in $\mathbb{E}^3$

Károly Böröczky
Eötvös Loránd University, Hungary

László Szabó
University of West Hungary, Hungary

A packing of unit balls in $\mathbb{E}^3$ is said to be a 12-neighbour packing if each ball is touched by 12 others. A 12-neighbour packing of unit balls can be constructed as follows. Consider a horizontal hexagonal layer of unit balls in which the centres of the balls are coplanar and each ball is touched by six others. Put on the top of this layer a second horizontal hexagonal layer of unit balls so that each ball of the first layer touches three balls of the second layer. The translation which carries the first layer into the second one, carries the second layer into a third one, and repeated translations of the same kind in both directions produce a packing of unit balls in which each ball has 12 neighbours. László Fejes Tóth conjectured that any 12-neighbour packing of unit balls in $\mathbb{E}^3$ is composed of such hexagonal layers. In September 2012 Thomas Hales posted a paper on the preprint server arXiv with a computer-assisted proof of this conjecture. The aim of this talk is to give a more geometric proof of the conjecture along a different line.

$SL(n, Z)$ intertwining and translation invariant Minkowski valuations on lattice polytopes

Károly J. Böröczky
Alfréd Rényi Institute of Mathematics, Hungary and Central European University, Hungary

We characterize translation invariant and either $SL(n, Z)$ contravariant or $SL(n, z)$ equivariant and Minkowski valuations on lattice polytopes. Along the way, we introduce the discrete Steiner point, which is the unique unimodular equivariant vector valued valuation on lattice polytopes.

This is joint work with Monika Ludwig.

Chirality in discrete structures

Marston Conder
University of Auckland, New Zealand

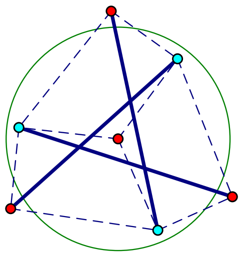
Symmetry is pervasive in both nature and human culture. The notion of chirality (or 'handedness') is similarly pervasive, but less well understood. In this lecture, given in celebration of Egon Schulte and Károly Bezdek's 60th birthdays, I will talk about discrete objects that have maximum possible rotational symmetry in their class, but are not 'reflexible'. The main examples are orientably-regular maps (and the associated Riemann surfaces), and abstract polytopes. Finite chiral polytopes of large rank are notoriously difficult to construct, but I will describe some new approaches (developed in joint work with a number of people) that provide some evidence that they are not quite as rare as once thought.

Bipartite graphs and distance geometry

ROBERT CONNELLY
Cornell University, United States

STEVEN GORTLER
Harvard University, United States

Given the lengths of the edges of a graph, one wants to find some configuration in $\mathbb{E}^d$ that realizes those lengths. In general this is a hard problem, but there are methods, using semi-definite programing, to decide when the graph can be realized with those edge lengths in some high dimensional Euclidean space. For the case of complete bipartite graphs, one can give a simple characterization of when this process produces a configuration in $\mathbb{E}^d$ and when it does not. When the configuration is symmetric, as in the Figure below, this determination is extremely simple.



On the geometry of linear optimization

ANTOINE DEZA
McMaster University, Canada

Finding good bounds on the maximal diameter $\Delta(d, n)$ of the vertex-edge graph of a polytope in terms of its dimension d and the number of its facets n is one of the basic open questions in polytope theory. The Hirsch conjecture, formulated in 1957 states that $\Delta(d, n)$ is at most $n - d$. While the conjecture was disproved by Santos in 2011, it is known to hold in small dimensions along with other specific pairs of d and n. However, the asymptotic behaviour of $\Delta(d, n)$ is not well understood: the best upper bound is quasi-polynomial. The behaviour of $\Delta(d, n)$ is not only a natural question of extremal discrete geometry, but is historically closely connected with the theory of simplex methods which, with the interior point methods, are the most successful algorithms for linear optimization. While simplex methods follow an edge path, interior point methods follow the central path. Within this framework, the curvature of a polytope, defined as the largest possible total curvature of the associated central path, can be regarded as the continuous analogue of its diameter. We present older and recent results dealing with the diameter and the curvature of polytopes. Based on joint works with Tamás Terlaky (Lehigh), Feng Xie (Microsoft), and Yuriy Zinchenko (Calgary).

Relaxed disk packing

Herbert Edelsbrunner
Institute of Science and Technology Austria, Austria

Motivated by biological questions, we study configurations of equal-sized disks in the Euclidean plane that neither pack nor cover. Measuring the quality by the probability that a random point lies in exactly one disk, we show that the regular hexagonal grid gives the maximum among lattice configurations.

This is joint work with Mabel Iglesias-Ham and Vitaliy Kurlin.

Beyond Voronoi: Generalized power diagrams, balanced k-means, and the representation of polycrystals

Peter Gritzmann
Technische Universität München, Germany

Based on a discrete convex maximization model we give an efficient algorithm for computing feasible generalized power diagams with near-optimal separation properties. Further, we show how this approach can be used to generalize the classical k-means algorithms from data analysis so that it becomes capable of handling weighted point sets and prescribed lower and upper bounds on the cluster sizes. (This part is joint work with S. Borgwardt and A. Brieden). Also we indicate how to handle the discrete inverse problem from material science to compute grain maps i.e., representations of polycrystals, based only on measured data on the volume, center and, possibly, moments of their grains. (This part is joined work with A. Alpers, A. Brieden, A. Lyckegaard and H. Poulsen)

Packings of Regular Pentagons

Thomas Hales
University of Pittsburgh, USA

Kuperberg and Kuperberg have described a double-lattice packing of regular pentagons. They conjecture it is the densest packing. This talk will present work in progress directed toward proving the optimality of the Kuperberg-Kuperberg double-lattice packing. This is work with Wöden Kusner.

Volume of convex hull of two bodies and related problems

Ákos G. Horváth
Budapest University of Technology and Economics, Hungary

We deal with some natural geometric problems concerning the volume of the convex hull of two "connecting" bodies. This problem follows from the elementary question: *How do we have to place two regular simplices with common centre in* $\mathbb{E}^3$ *as to maximize the volume of their convex hull?*

We examine generally the volume of the convex hull of two congruent copies of a convex body in Euclidean n-space, under some subsets of the isometry group of the space. We also investigate the behaviour of the volume that the convex hull of two congruent and intersecting simplices in Euclidean n-space can have.

To find the maximal volume polyhedra with a prescribed property and inscribed in the unit sphere is also an interesting question with such hard (and generally unsolved) special case as *to find the maximal volume polyhedra with given number of vertices inscribed in the unit sphere.*

In this talk we collect some methods, results and open problems connecting to the above questions, respectively.

On polytopes and maniplexes

Isabel Hubard and Jorge Garza
Instituto de Matemáticas, UNAM, Mexico

In this talk we'll discuss abstract polytopes seen as maniplexes, in particular we present necessary and sufficient conditions for a maniplex to be a polytope. Depending on time, we shall see some results of polytopes from the maniplex point of view.

Hexagonal extensions of toroidal hypermaps

Dimitri Leemans
University of Auckland

Hypertopes are thin residually connected incidence geometries. These include polytopes (when the corresponding diagram is a string) and most hypermaps. Recently, with Maria Elisa Fernandes and Asia Ivić Weiss, we have characterised the automorphism groups of regular and chiral hypertopes. Toroidal hypertopes of rank three are divided in the following families: the toroidal maps $\{3,6\}_{(b,c)}$, $\{6,3\}_{(b,c)}$, $\{4,4\}_{(b,c)}$ and the hypermaps $(3,3,3)_{(b,c)}$ with $(b,c) \neq (1,1)$. In this talk, we will discuss the classification of hexagonal extensions of toroidal hypermaps.These are rank four hypertopes with vertices, edges, faces and facets, such that facets are toroidal hypermaps and in which a specific rank two residue is an hexagon. One example is the polytopes of type $[6,3,6]$. We will show some examples of regular and chiral such hypertopes. This in joint work with Maria Elisa Fernandes and Asia Ivić Weiss.

Vertex index of symmetric convex bodies

Alexander Litvak
University of Alberta, Canada

We discuss several results on the vertex index of a given d-dimensional centrally symmetric convex body, which, in a sense, measures how well the body can be inscribed into a convex polytope with small number of vertices. This index is closely connected to the illumination parameter of a body, introduced earlier by Karoly Bezdek, and, thus, related to the famous conjecture in Convex Geometry about covering of a d-dimensional body by 2^d smaller positively homothetic copies. We provide estimates of this index and relate the lower bound with the outer volume ratio. We also discuss sharpness of the bounds, providing examples. The talk is based on joint works with K. Bezdek and E.D. Gluskin.

Results and problems from Minkowski geometry

Horst Martini
TU Chemnitz, Germany

The geometry of normed or Minkowski spaces (i.e., of finite dimensional real Banach spaces) can be considered as an extension of Classical Convexity (since the unit ball can be an arbitrary convex body centered at the origin) and a ßubcaseöf Finsler Geometry (local point of view) and of real Banach space theory (finite dimensional considerations only). The axiomatic foundations were developed by H. Minkowski more than hundred years ago, and the large amount of results proved until now (and connected with names like H. Busemann, M. M. Day, J. J. Schaeffer, H. Guggenheimer, B. Gruenbaum, V. Klee, A. C. Thompson, and many others) was only partially systematized in the literature. Modern developments show how this area fruitfully starts to combine with modern fields like Discrete Differential Geometry, Computational Geometry, and Discrete Geometry. But also related new ways are gone in more traditional disciplines, like Functional Analysis, Approximation Theory, Convexity, Classical Differential Geometry, even Elementary Geometry, and others. After some introductory part, we will present examples for some of these modern developments. These will refer to the following partial fields:

1) *Convexity in Minkowski spaces:* Here we will concentrate on special classes of convex bodies, e.g. on bodies of constant width. Well known generalizations of this class are the families of complete bodies (any proper superset of a complete body has larger diameter) and reduced bodies (any convex proper subset of a reduced body has smaller minimal width). It turns out that the geometry of reduced and complete bodies is much richer in Minkowski spaces than in the Euclidean subcase.

2) *Discrete Geometry in Minkowski spaces:* Here we will discuss the extension of the famous Fermat-Torricelli problem from the Euclidean subcase to Minkowski spaces. It is shown how the geometry of the unit ball influences the shape of the solution set, and related results are meanwhile even extended to gauges (i.e., to planes whose convex unit ball no longer has to be centrally symmetric). Second, we will present results related to Lebesgue's universal-cover problem.

3) *Classical curves in Minkowski planes:* Here we will discuss the geometry of multifocal ellipses and Cassini curves in normed planes, again mentioning also extensions to gauges.

4) *Elementary Geometry in normed planes:* As typical examples, we will discuss extensions of Titeica's three-circles theorem and Miquel's theorem to normed planes. Based on this, we will show how classical Euclidean theorems are getting "poorer" when switching to normed planes (e.g., the Feuerbach circle remains only as a six-point circle in general normed planes).

Just a snip

Barry Monson
University of New Brunswick, Canada

Leah Berman and Gordon Wiiliams
University of Alaska - Fairbanks, United States

Deborah Oliveros
UNAM - Querétaro, Mexico

There are several ways to symmetrically truncate a regular convex polytope. Here we cut off each corner of a regular n-simplex Δ_n one-third of the way in along each edge. The resulting *truncated simplex* $\mathcal{T}_n$ is uniform and has the same symmetry group Sym_{n+1} as Δ_n (except when $n = 2$: $\mathcal{T}_2$ is a regular hexagon).

The *minimal regular cover* of $\mathcal{R}_n$ is an abstract regular polytope. For example, the ordinary truncated tetrahedron $\mathcal{T}_3$ is covered by the regular toroidal map $\{6, 3\}_{(2,2)}$. Determining $\mathcal{R}_n$ really amounts to understanding the *monodromy group* M_n of $\mathcal{T}_n$.

In fact, $M_n \simeq \mathrm{Sym}_{n+1} \times \mathrm{Sym}_n$, and reinterpretations of $\mathcal{R}_n$ can be found in several projects in which our friend Egon Schulte had a hand: it is a certain mix; a particular universal amalgamation; a graphicahedron; the dual of an earlier regular incidence-polytope.

We discus all this and end with a look some new familes of regular poytopes arising from some hocus-pocus applied to a Coxeter diagram related to $\mathcal{R}_n$.

Kneser transversals

Luis Montejano
National University of Mexico at Queretaro, Mexico

Let X be a set of n points in $\mathbb{R}^d$. We are interested in studying whether there is a transversal $(d - \lambda)$-plane to the convex hulls of all k-subsets of X. In particular, we would like to know how small n must be to ensure that there will always be such a transversal. This problem has interesting and deep connections with classic problems as Rado's central affine plane problem and the calculation of the chromatic number of Kneser hypergraphs.

Helical chiral polyhedra

Daniel Pellicer, Javier Bracho, Isabel Hubard
UNAM, Mexico

In this talk a *polyhedron* is a discrete geometric structure consisting of vertices (points), edges (line segments between pairs of points) and faces (cycles or infinite paths) satisfying the diamond condition and strong flag-connectivity. A polyhedron is said to be *chiral* if it has symmetries that cyclically permute the vertices around a base face f, and symmetries that cyclically permute the faces around a base vertex v contained in f, but it does not contain a non-trivial symmetry fixing both v and f. Polyhedra containing all previously mentioned symmetries are called *regular*.

The faces of a chiral polyhedron in the Euclidean or projective space may be planar, skew or helical. The vertices of a helical face are obtained as the orbit of a screw motion in the Euclidean space $\mathbb{E}^3$ (and hence the faces are infinite), and of a double rotation in the real projective space $\mathbb{P}^3$.

In 2005 Schulte classified all chiral polyhedra in $\mathbb{E}^3$, including three families of polyhedra with helical faces. The polyhedra on these families can be alternatively constructed from three previously known regular polyhedra by a variation of Wythoff's construction.

In this talk we review these topics and compare it with the current status of the determination of chiral polyhedra with helical faces in $\mathbb{P}^3$.

Orientable vs. oriented

Tomaž Pisanski
University of Primorska and University of Ljubljana, Slovenia

Sometimes one is tempted to mix up the notions of "oriented map" and "orientable map", "oriented surface" and "orientable surface", etc. The purpose of this talk is to point out a clear difference between the concepts of "oriented structure" and "orientable structure" within the framework of generalized action graphs by identifying certain relationships between the two closely related concepts. Our generalized action graphs generalize the concept of "action graphs", introduced by Malnič in 2002 and is identical to a structure that can be imposed on certain regular graphs and was considered by Exoo, Jajcay and Širáň in 2013 in search of Cayley cages.

The degree-diameter problem for vertex-transitive graphs and finite geometries

JOZEF ŠIRÁŇ
Open University, United Kingdom and Slovak University of Technology, Slovakia

The problem in the title is to find the largest order of a vertex-transitive graph of a given degree and diameter. A natural upper bound on this order is the well-known Moore bound. We show how one can use generalised triangles and quadrangles to produce infinite families of vertex-transitive (even Cayley) graphs of diameter 2 and 3 which, in some sense, asymptotically approach the corresponding Moore bounds, and discuss related questions.

Approximate Steiner trees

KONRAD SWANEPOEL
London School of Economics, United Kingdom

A Steiner minimal tree of a given set N of terminals in d-dimensional Euclidean space is a shortest tree that interconnects the set N. We allow the tree to have additional vertices, called Steiner points. At each Steiner point, three edges come together at 120 degrees. In the plane, Steiner minimal trees can be constructed by ruler and compass, but in higher dimensions this is not true any more, and exact calculations are replaced in practice by numerical approximations. Rubinstein, Weng and Wormald (2006) studied the worst-case error in the length of so-called epsilon-approximate Steiner trees, where all angles at Steiner points are within epsilon of 120 degrees. We give an overview of what is known about this error, including some new results.

This is joint work with Charl Ras (Mathematics, University of Melbourne) and Doreen Thomas (Mechanical Engineering, University of Melbourne).

Bounding minimal solid angles of polytopes

Arseniy Akopyan
Institute of Science and Technology Austria, Austria

In this article we study the following question: What can be the measure of the minimal solid angle of a simplex in $\mathbb{R}^d$? We show that in dimensions three and four it is not greater than the solid angle of the regular simplex. We also study a similar question for trihedral and dihedral angles of polyhedra compared to those of regular solids.

This is a joint work with Roman Karasev (Moscow Institute of Physics and Technology, Russia).

Pseudoachromatic and connected-pseudoachromatic indices of the complete graph

Gabriela Araujo-Pardo, C. Rubio Montlel
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico.

A *complete k-coloring* of a graph G is a (not necessarily proper) k-coloring of the vertices of G, such that each pair of different colors appears in an edge. A complete k-coloring is also called *connected* if each color class induces a connected subgraph of G. The *pseudoachromatic index* of a graph G, denoted by $\psi'(G)$, is the largest k for which the line graph of G has a complete k-coloring, analogously the *connected-pseudoacrhomatic index* of G, denoted by $\psi'_c(G)$, is the largest k for which the line graph of G has a connected and complete k-coloring.

In this paper we study these two parameters for the complete graph K_n. First, we prove that, when q is a power of 2 and $n = q^2 + q + 1$, the pseudoachromatic index $\psi'(K_n)$ of the complete graph K_n is at least $q^3 + 2q - 3$; which improves the bound $q^3 + q$ given by Araujo, Montellano and Strausz [J Graph Theory 66 (2011), 89–97]. Our main contribution is to improve the linear lower bound for the connected pseudoachromatic index given by Abrams and Berman [Australas J Combin 60 (2014), 314–324] and provide an upper bound, these two bounds prove that for any integer $n \geq 8$ the order of $\psi'_c(K_n)$ is $n^{3/2}$.

Remark: The colorations that induce the lower bounds are given using the structure of Projectives Planes.

Existence of closed billiard trajectories in "acute-angled" bodies

Alexey Balitskiy
Moscow Institute of Physics and Technology, Russia

We will discuss several recent results about billiards in convex bodies. We mostly use the approach of Daniel Bezdek and Károly Bezdek, sometimes adapted to the case of arbitrary Minkowski norm.

In particular, we study the question of existence of periodic billiard trajectory in a non-smooth convex body K in $\mathbb{R}^n$. For the non-smooth case the strongest result at this moment is the existence of billiard trajectory in triangle with angles not greater then 100°. We consider "generalized" trajectories, which may bounce at non-smooth points of body boundary. If for every point $q \in \partial K$ there exists $\varphi < \pi/2$ such that for all points $a, b \in K$ holds inequality $\widehat{aqb} \leq \varphi < \frac{\pi}{2}$ we say that point q satisfies the *acuteness condition.* We mainly discuss "acute-angled" convex bodies, and show that minimal (by length) "generalized" trajectory should be "classical". In some particular cases "acuteness" condition may be weakened.

This is joint work with Arseniy Akopyan.

Geometric movable configurations

Leah Wrenn Berman
University of Alaska Fairbanks, United States

A geometric k-configuration is a collection of points and straight lines, typically in the Euclidean plane, such that every point lies on k lines and every line passes through k points. A geometric configuration is *movable* if it admits a continuous family of realizations fixing four points in general position but moving at least one other point. It is easy to construct movable geometric configurations which have little or no non-trivial geometric symmetry, but it is considerably more challenging to produce movable configurations with high degrees of rotational or dihedral symmetry. This talk will provide an overview of a number of constructions for movable geometric k-configurations with a high degree of geometric symmetry, including new constructions that produce the first known examples of symmetric movable 5-configurations.

On the volume of boolean expressions of large congruent balls

Balázs Csikós
Eötvös Loránd University, MTA Alfréd Rényi Institute of Mathematics, Hungary

Let $P_1, \ldots, P_N$ be points in the Euclidean space E^n. It is known that that for large $r > 0$, the volume of the union of the balls $B_i = B(P_i, r)$ is equal to $\omega_n(r^n + \frac{n}{2}w(K)r^{n-1} + O(r^{n-2}))$, where ω_n is the volume of the n-dimensional unit ball, K is the convex hull of the points $P_1, \ldots, P_N$, and $w(K)$ is the mean width of K. Igors Gorbovickis showed that we have $\mathrm{vol}(\bigcap_{i=1}^N B_i) = \omega_n(r^n - \frac{n}{2}w(K)r^{n-1} + O(r^{n-2}))$ for the volume of the intersection of the balls B_i. The mean width that appears in the coefficient of r^{n-1} in both cases has the following monotonicity property: If $Q_1, \ldots, Q_N$ are points in E^n such that $d(P_i, P_j) \geq d(Q_i, Q_j)$ for all $1 \leq i, j \leq N$, and $\hat{K}$ is the convex hull of the points $Q_1, \ldots, Q_N$, then $w(K) \geq w(\hat{K})$. We are going to generalize these results to bodies that can be obtained from the balls by the Boolean operations $\cup$, $\cap$, and $\setminus$. We find a formula for the volume of such bodies modulo $O(r^{n-2})$ and give geometrical description of the coefficient of r^{n-1} in the formula. We also study monotonicity properties of the coefficient of r^{n-1}.

Approximations of round convex bodies in the plane and in higher dimensions

Ferenc Fodor
University of Szeged, Hungary

We will consider some classical approximation problems for convex bodies both in the plane and in higher dimensions, and investigate various generalisations of these problems for bodies that have a certain degree of roundness. Our main focus will be on convex bodies that are intersections of congruent closed balls; such objects are often called spindle convex or hyperconvex. In our review of the recent progress in this topic, we will explore some approximation properties of round convex bodies that are analogous to the linearly convex case and we will also encounter some surpsising facts that have no linear analogue.

A large part of the results presented in this talk are joint with G. Fejes Tóth, P. Kevei, Á. Kurusa and V. Vígh.

Symmetries of Monocoronal Tilings

Dirk Frettlöh
Technische Fakultät, Univ. Bielefeld, Germany

Alexey Garber
Department of Mathematics, UT Brownsville, Texas, United States

The study of tilings using a single tile shape, resp. vertex shape, resp.... has a long history. Whereas a classification of tilings of the Euclidean plane by pairwise congruent tiles is still incomplete, a complete classification of tilings of the Euclidean plane with pairwise congruent vertex corona has been established by the two authors in 2014. The vertex corona of a vertex in a tiling is the patch of all tiles adjacent to this vertex. A tiling where all vertex coronae are congruent (reflections allowed) is called *monocoronal*. A tiling where all vertex coronae are directly congruent (reflections forbidden) is called *monocoronal wrt rigid motions*. In Frettlöh and Garber 2015+ (arXiv:1402.4658) the following results were established:

- Every monocoronal tiling wrt rigid motions (no reflections allowed) has as its symmetry one out of 12 wallpaper groups. In particular, any such tiling is crystallographic.
- Every monocoronal tiling (reflections allowed) is either 1-periodic, or its symmetry is one out of 16 wallpaper groups. If such a tiling is 1-periodic then its symmetry group is one out of four frieze groups.

Furthermore some partial results on monocoronal tilings in higher dimensional Euclidean and hyperbolic space were obtained. For instance, there are monocoronal tilings wrt rigid motions in d-dimensional Euclidean space for $d \geq 4$ with a wide range of symmetry groups; from trivial to crystallographic. The same is true for monocoronal tilings in d-dimensional hyperbolic space for $d \geq 2$.

Five-dimensional Dirichlet-Voronoi parallelohedra

Alexey Garber
The University of Texas at Brownsville, United States

In this talk we will report about full classification of combinatorially different five-dimensional Dirichlet-Voronoi parallelohedra for lattices.

The classification of affinely different Delone triangulations (L-type domains) can be done using Voronoi's second reduction theory. Dimensions 3 and 4 can be done without using the reduction theory, but already in dimension 5 it plays an important for classification. The classification of five-dimensional L-type domains was made by E. Baranovskii and S. Ryshkov in 1973. They found 221 different triangulations, but later P. Engel in 1998 found that they missed one triangulation.

In this talk we will show an extension of the Voronoi's reduction theory to find all affinely non-equivalent lattice Delone decompositions and combinatorially different Dirichlet-Voronoi parallelohedra in arbitrary dimension and present our computational results in dimension 5.

Our main result is the following

Theorem. *There are* 110244 *affine types of lattice Delone triangulations and* 110244 *combinatorial types of Dirichlet-Voronoi parallelohedra in dimension* 5*.*

This is a joint work with M. Dutour Sikirić, A. Schürmann, and C. Waldmann.

Polyhedral realization of regular maps of genus at least 2

Gábor Gévay
University of Szeged, Hungary

We review the known polyhedral realizations (embeddings) in Euclidean 3-space of regular maps with genus $g \geq 2$. We discuss a most recently published result, an embedding of the Fricke–Klein map of genus 5, which is known as the Grünbaum polyhedron (joint result with Egon Schulte and Jörg M. Wills). We also present a new infinite series of regular maps of Schläfli type $\{6, 4\}$, and their duals; these maps are obtained from the corresponding Petrie-Coxeter apeirohedra by a factorization procedure. No embedding in Euclidean 3-space is known for these maps; instead, we outline how they can be embedded in 6-space.

Colorings, monodromy, and impossible triangulations

Ivan Izmestiev
Freie Universitat Berlin, Germany

If a triangulation of the sphere has only two vertices of odd degree, then these cannot be adjacent. We prove this curious fact by studying the colouring monodromy, an obstruction to the existence of a vertex colouring. The monodromy also leads to a construction of equivelar surfaces of high genus.

Similar in the spirit is the following theorem. There is no triangulation of the torus with all vertex degrees 6, except for one of degree 5 and one of degree 7. This is proved by studying the rotation monodromy of the geometric structure obtained by making each triangle of the triangulation to an equilateral euclidean triangle. The second result is a joint work with Kusner, Rote, Springborn, and Sullivan.

Cassini sets in generalized Minkowski spaces

Thomas Jahn, Horst Martini
Technische Universität Chemnitz, Germany

Christian Richter
Friedrich Schiller University Jena, Germany

The class of *Cassini curves*, which has applications in several fields of mathematics and physics, is a popular object of investigation in classical curve theory. Based on the definition as the geometric locus of points whose product of distances from two fixed points (called *foci*) is constant, we have a closer look at a twofold generalization of these curves by transferring this concept to generalized Minkowski spaces, that is, finite-dimensional vector spaces endowed with an abstract notion of distance provided by a gauge, and to the multifocal case. In particular we focus on starshapedness and connectedness results regarding the sets of points whose product of distances from the foci does not exceed a certain level.

Integers, modular groups, and hyperbolic space

Norman W. Johnson
Wheaton College, United States

In each of the normed division algebras over the real field $\mathbb{R}$ — namely, $\mathbb{R}$ itself, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ — certain elements can be characterized as *integers*. An integer of norm 1 is a *unit*. In a *basic system* of integers the units span a 1-, 2-, 4-, or 8-dimensional lattice, the points of which are the vertices of a regular or uniform Euclidean honeycomb. A *modular group* is a group of linear fractional transformations whose coefficients are integers in some basic system. In the case of the octonions, which have a nonassociative multiplication, such transformations form a *modular loop*. Each real, complex, or quaternionic modular group can be identified with a subgroup of a Coxeter group operating in hyperbolic space of 2, 3, or 5 dimensions.

Symplectic ideas in convex geometry

Roman Karasev
Moscow Institute of Physics and Technology, Moscow, Russia

I will survey recent works where the techniques of symplectic geometry helped to prove some results or provide the valuable intuition in some problems of convex and discrete geometry. This started from the work of Artstein-Avidan and Ostrover relating the mysterious invariants called “symplectic capacities” to much more elementary invariants of billiards in convex bodies.

After that, in subsequent works with my participation the billiards were studied more deeply and the famous conjectures of Mahler and Bang in convex geometry were related to certain conjectures in symplectic geometry. We do not always find the necessary symplectic machinery for our convexity problems; but we benefit from the general point of view provided by symplectic geometry and see the “right questions” to ask in convex and discrete geometry.

Quantitative covering of convex bodies

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The Gohberg-Markus-Hadwiger Covering Conjecture (equivalently, the Boltyanski-Hadwiger Illumination Conjecture) is one of the oldest and most important open questions in discrete geometry. Due to widespread interest in this problem, a number of quantities related to the homothetic covering and illumination of convex bodies have been introduced. For instance, Bezdek defined the illumination parameter, Swanepoel the covering parameter and more recently, the authors introduced the covering index of a convex body. Intuitively, the covering index measures how well a convex body can be covered by a relatively small number of positive homothets having a relatively small homothety ratio not exceeding one-half. The authors also showed that restricting the homothety ratios to be no greater than one-half endows the covering index with some interesting properties. However, it is natural to ask what happens if we remove this restriction? Another matter concerns how to define fractional analogues of the covering index. In this talk, we address these questions and further investigate the quantitative covering of convex bodies.

The robustness of convex bodies

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Budapest University of Technology, Hungary

We examine the minimal magnitude of perturbations necessary to change the number N of static equilibrium points of a convex solid K. We call the normalized volume of the minimally necessary truncation *robustness* and we seek shapes with maximal robustness for fixed values of N. While the *upward* robustness (referring to the increase of N) of smooth, homogeneous convex solids is known to be zero, little is known about their *downward* robustness. The difficulty of the latter problem is related to the coupling (via integrals) between the geometry of the hull $\operatorname{bd} K$ and the location of the centre of gravity G. Here we first investigate two simpler, decoupled problems by examining truncations of $\operatorname{bd} K$ with G fixed, and displacements of G with $\operatorname{bd} K$ fixed, leading to the concept of *external* and *internal* robustness, respectively. In dimension 2, we find that for any fixed number $N = 2S$, the convex solids with both maximal external and maximal internal robustness are regular S-gons. We also show that in the decoupled problems, 3-dimensional regular polyhedra have maximal internal robustness, however, only under additional constraints. Finally, we prove results for the full problem in case of 3 dimensional solids. These results appear to explain why monostatic pebbles (with either one stable, or one unstable point of equilibrium) are found so rarely in Nature. This project is a joint work with G. Domokos.

Vertex-transitive polyhedra, their maps and quotients

Undine Leopold
Technische Universität Chemnitz, Germany

Since Grünbaum and Shephard's 1984 article on "Polyhedra with Transitivity Properties" the complete classification of vertex-transitive polyhedra of genus $g \geq 2$ in Euclidean 3-space has proven to be a difficult problem despite its rigid setting. Here, polyhedra are face-to-face tessellations of closed, connected, orientable, embedded surfaces by simple, plane polygons. Seven examples are known, and it is also known that the symmetry groups must be among the rotation groups of the Platonic solids. In my dissertation at Northeastern University the tetrahedral case was solved. In this talk, I will focus on ways of obtaining geometric insight into the problem by examining geometric and topological aspects of the underlying maps and their quotients by the symmetry group.

On the volume product

Endre Makai, Jr.
Alfréd Rényi Institute of Mathematics, Hungary

Let $K \subset \mathbf{R}^n$ be an 0-symmetric convex body, and let K^* denote its polar body $\{y \in \mathbf{R}^n \mid \forall x \in K \ \langle y, x\rangle \leq 1\}$. Fixing $V(K)$, we are interested in the minimal and maximal values of $V(K^*)$.

The *volume product* of K is $V(K)V(K^*)$ (where $V(\cdot)$ denotes volume). This is invariant under non-singular linear transformations of K (hence is a dimensionless quantity), and hence has a positive lower bound and a finite upper bound, whose determination is equivalent to the determination of the minimal and maximal values of $V(K^*)$, if $V(K)$ is any given number. The upper bound is attained exactly for the ellipsoids. The conjectured lower bound is $4^n/n!$ which is attained for the parallelotope and for the cross-polytope, but for $n \geq 4$ also for a large number of other bodies (called Hanner-Hansen-Lima polytopes). It is conjectured that the minimum is attained just for these polytopes.

The problem has an asymmetric variant as well.

Based on our recent paper with K. J. Böröczky, M. Meyer, S. Reisner I will sketch some of the best upper and lower estimates known up to now.

Geometry of knots, links and polyhedra

Alexander Mednykh
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Laboratory of Quantum Topology, Chelyabinsk, Russia

We investigate the structure of fundamental polyhedra for knots and links modelled on the hyperbolic, spherical and Euclidean spaces. The existence theorems for fundamental polyhedra are given and the volume formulae are presented.

Classification of fundamental domains for cocompact plane groups

Emil Molnár
Budapest University of Technology and Economics, Hungary

Zoran Lučić, Nebojša Vasiljević
University of Belgrade, Serbia

H. Poincaré (1882) attempted to describe a plane crystallographic group in the Bolyai-Lobachevsky hyperbolic plane $\mathbb{H}^2$ by appropriate fundamental polygon. This initiative he extended also to space. B. N. Delone (Delaunay) in 1960's refreshed this very hard topic for Euclidean space groups by the so-called stereohedron problem: To give all fundamental domains for a given space group; with few partial results.

A. M. Macbeath (1967) completed the initiative of H. Poincaré in classifying the 2-orbifolds by giving each with a signature. That is by a base surface with orientable or non-orientable genus; by some singular points on it, as rotational centers with given periods; by some boundary components, in each with given dihedral corners. All these are characterized up to equivariant isomorphism, also reported in this talk.

There is a nice curvature formula that describes whether the above (good) orbifold, i.e. cocompact plane group (with compact fundamental domain) is realizable either in the sphere $\mathbb{S}^2$, or in the Euclidean plane $\mathbb{E}^2$, or in the hyperbolic plane $\mathbb{H}^2$, respectively.

Our initiative in 1990's was to combine the two above descriptions; namely, how to give all the combinatorially different fundamental domains for any above plane group. Z. Lučić and E. Molnár completed this by a graph-theoretical tree enumeration algorithm. That time N. Vasiljević implemented this algorithm to computer (program COMCLASS), of super-exponential complexity, by certain new ideas as well. In the time of the Yugoslav war we lost our manuscript, then the new one has been surprisingly rejected (?!).

Now we have refreshed our manuscript to submit again. Here you are presented a report on it.

Sperner type lemmas and related topics

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In this talk we consider several generalizations of the classic Sperner lemma. The Sperner, Tucker and Ky Fan lemmas are combinatorial analogs of the Brouwer and Borsuk - Ulam (BUT) theorems with many useful applications. These classic lemmas are concerning labelings of triangulated discs and spheres. We show that discs and spheres can be substituted by large classes of manifolds with or without boundary. We also present a generalization of Sperner's lemma where instead of triangulations are considered quadrangulations.

In 1996 Shashkin proved a version of Fan's lemma, which is a combinatorial analog of the odd mapping theorem (OMT). We will discuss generalizations of these lemmas for BUT–manifolds. Our proofs rely on a generalization of the OMT and on the lemma about the doubling of manifolds with boundaries that are BUT–manifolds.

With any cover of a space T we associate certain homotopy classes of maps T into n–spheres. These homotopy invariants can be considered as obstructions for extensions of covers of a subspace A to a space X. We using these obstructions for generalizations of the classic KKM and Sperner lemmas.

On homothetic copies of a convex body

Márton Naszódi
EPFL, Switzerland & ELTE, Hungary

We discuss some questions concerning families of intersecting and touching positive homothets of a convex body. One example is the Bezdek–Pach Conjecture, according to which the maximum number of pairwise touching homothetic copies of a convex body in Euclidean d-space is 2^d. Another is a question of Füredi and Loeb: What is the largest number of balls (of possibly different radii) in a Minkowski space that pairwise intersect but none of whom contains the center of another.

Exploring the concept of perfection in 3–hypergraphs

Natalia García-Colín, Deborah Oliveros and Amanda Montejano
UNAM, Mexico

By exploring the concept of perfection in uniform m–hypergraphs, as a natural extension of perfection in graph theory; A natural question rises: Is a m–hypergraph H perfect, if for every subhypergraph H', $\chi(H') = \lceil \frac{\omega(H')}{m} \rceil$, where $\chi(H')$ and $\omega(H')$ are the chromatic and clique number of H', respectively?

One of the classical examples of perfect graphs is the family of comparability graphs, in this talk we introduce the concept of comparability 3–hypergraphs (those that can be transitively oriented) with the aim of proving that these are not perfect according to the natural definition. More explicitly, we exhibit three different subfamilies of comparability 3–hypergraphs which show different behaviors in respect to the relationship between the chromatic number and the clique number.

Flag-transitive biplanes, what do we know?

Eugenia O'Reilly-Regueiro
IMUNAM, Mexico

Biplanes are $(v, k, 2)$- symmetric (balanced incomplete block) designs (SBIBDs). There seems to be a big difference between $(v, k, 1)$-symmetric designs (which are projective planes), and (v, k, λ)-symmetric designs with $\lambda > 1$. It has long been conjectured that for any given value of $\lambda > 1$, only finitely many (v, k, λ)-symmetric designs exist. The first case to consider is of course $\lambda = 2$, that is, biplanes. A flag in a symmetric design is an incident point-block pair, and a group of automorphisms of a design is *flag-transitive* if it is transitive on the flags of the design. In this talk we will review the (incomplete) classification of flag-transitive biplanes, as well as some results in the open case. This is joint work with Cheryl Praeger and Patricio Ricardo García-Vázquez.

Polytopes with preassigned automorphism groups

Egon Schulte
Northeastern University, USA

Gordon Ian Williams
University of Alaska Fairbanks, USA

We show constructively that every finite group can interpreted as the automorphism group of a finite spherical abstract polytope. In addition, we show that every finite group arises as the automorphism group of a convex polytope.

The twist operator on maniplexes

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Northern Arizona University, United States

Isabel Hubard, Daniel Pellicer
UNAM, Mexico

If $\mathcal{M} = (\Omega, [r_0, r_1, r_2, ..., r_n])$ is an n-maniplex ('maniplex' is a generalization of 'map' to higher dimensions) which is orientable and rotary (i.e., it has all possible orientation-preserving symmetries), and w is a suitably chosen word in generators of index no more than $n-2$, then we can form a new maniplex $\mathcal{N} = T_w(\mathcal{M})$ by replacing r_n with wr_n for half the flags and $w^{-1}r_n$ for their neighbors. The new maniplex has the same facets as the original, but the connections between facets are twisted by w.

We will show examples of this construction and, perhaps, show how it might be used to construct chiral maniplexes (which might be polytopal) of all dimensions.

Maximality properties of rational lattice-free polyhedra

Jan Krümpelmann
Otto-von-Guericke University of Magdeburg, Germany

Let $d, s \in \mathbb{N}$ and let $\mathcal{P}(\frac{1}{s}\mathbb{Z}^d)$ be the family of all d-dimensional polyhedra P such that P is the convex hull of $P \cap \frac{1}{s}\mathbb{Z}^d$. We say that $P \in \mathcal{P}(\frac{1}{s}\mathbb{Z}^d)$ is lattice-free if the interior of P does not contain points of $\mathbb{Z}^d$. For a set $X \subseteq \mathbb{R}^d$, a lattice-free polyhedron is called X-maximal if for every point $x \in X \setminus P$, the convex hull of P and x is not lattice-free. Averkov, Wagner and Weismantel raised the question of characterizing the pairs (d, s) such that every lattice-free polyhedron $P \in \mathcal{P}(\frac{1}{s}\mathbb{Z}^d)$ is $\mathbb{Z}^d$-maximal if and only if P is $\mathbb{R}^d$-maximal. Nill and Ziegler answered the question for $d \geq 4$, where equivalence does not hold for any s, while equivalence trivially holds for $d = 1$. We complete the classification by proving that equivalence holds for $d = 3$ and $s = 1$ as well as $d = 2$, $s \in \{1, 2\}$ and does not hold for all other pairs (d, s) where $d \in \{2, 3\}$. This is joint work with Gennadiy Averkov and Stefan Weltge (both of Otto-von-Guericke University of Magdeburg, Germany).

On contact numbers of totally separable unit sphere packings

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$\mathbb{E}^d$ is the d-dimensional Euclidean space. The **contact graph** of any finite packing of *unit balls* (having pairwise disjoint interiors) is the graph whose vertices are the balls and two vertices are adjacent if they touch each other. The *number of edges* of this graph is the **contact number**. The *maximum number of edges* that a contact graph of n unit balls can have is denoted by $c(n,d) \quad (n > 1,\, d \geq 2)$. Clearly $c(n,d) \leq dn$. A packing is **totally separable** is any two balls can be separated by a *hyperplane* which is disjoint from the interior of each ball in the packing.

$c_{\mathbb{Z}}(n,d)$ denotes the largest possible contact number of all totally separable packings of n unit balls in the case every center of them is a lattice point of the integer lattice $\mathbb{Z}^d$. It is known that $c_{\mathbb{Z}}(n,2) = c(n,2) = \lfloor 2n - 2\sqrt{n} \rfloor$ for all $n > 1$. We do not know any explicit formula for $c_{\mathbb{Z}}(n,3)$, but we know $c_{\mathbb{Z}}(n,3) = 3n - 3n^{\frac{2}{3}} + o(n^{\frac{2}{3}})$ as $n \to +\infty$. Clearly, $c_{\mathbb{Z}}(n,3) \leq c(n,3)$ for all $n > 1$. So, one may wonder whether $c_{\mathbb{Z}}(n,3) = c(n,3)$ for all $n > 1$?

Theorem 1 $c_{\mathbb{Z}}(n,d) \leq \lfloor dn - dn^{\frac{d-1}{d}} \rfloor$ *for all* $n > 1$ *and* $d \geq 2$.
This is sharp for $d = 2$, $n > 1$, for $d \geq 3$, $n = k^d$, $k > 1$, but not for $d = 3$ and $n = 5$.

Theorem 2 $c(n,d) \leq \left\lfloor dn - n^{\frac{d-1}{d}} \Big/ 2d^{\frac{d-1}{2}} \right\rfloor$ *for all* $n > 1$ *and* $d \geq 4$.

Theorem 3 $c(n,3) < \lfloor 3n - 1.346 n^{\frac{2}{3}} \rfloor$ *for all* $n > 1$.

A rephrasing of edge colouring by local charts and orientability

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A graph or more generally a multigraph can be interpreted as a family of stars which adequately intersect on certain edges, so as to generate a global adjacency structure. An edge colouring can be read as an injective assignment of colours to each star, enjoying a "compatibility" property on adjacent vertices. This makes an edge colouring in some sense similar to a differentiable atlas on a manifold. In the case of simple graphs, the distinction between class 1 and class 2 (from Vizing's theorem) is analogous to the distinction between orientable and non-orientable atlases. In particular, small 3-critical graphs are the result of an identification of extremal edges or vertices which corresponds to the topological identification yielding the Möbius strip from the rectangular strip. In the present talk we revisit Jacobsen's classification of small 3-critical graphs, from this point of view.